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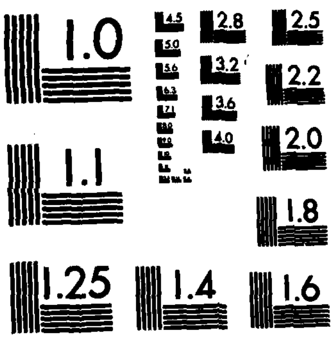

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The principal result is a complete theory for the pratical design of minimum sensitive linear feedback compensators. Sufficient conditions are developed from new theorems relating conjugate points to the positive definiteness and controllability of the accessory minimum problem. The advantages of the minimum sensitive compensator relative to least square parameter estimators are discussed. An example illustrates the improved sensitivity characteristics of the compensator as compared to model following and regulating controls.

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NOMENCLATURE

Symbol	*Description*
$\underline{a}$	Vector
A	Matrix or operator
A^*	Adjoint of A
A^T	Transpose of A
C^1	Space of continuously differentiable functions
$\Vert\underline{a}\Vert$	Norm of $\underline{a}$
R^n	Euclidean n space
st	Such that
WRT	With respect to
a.e.	Almost everywhere
μ	Lebesgue measure
$\forall$	For all
$\exists$	There exists
$\subset$	Contained in
$\in$	A member of
$\cup$	Union
$\nabla_s H$	Gradient of H WRT s

Chapter 1

Introduction

1.1 Problem Statement

The dynamical behavior of many physical processes can be described by differential equations. The ability of such a description to correctly predict the actual system response is directly related to the accuracy of the mathematical model. Errors in the model can result from unsatisfactory initial approximations and from actual component variations after the model has been produced. When a differential equation is used for controller design, the resulting control may greatly depend on various parameters in the model. If these parameters remain at their design values, the control input to the actual process will produce the desired output. However, if the actual system deviates from the model, the desired output may not be realized.

Modeling accuracy is particularly important in the practical application of optimal control theory. Here it is assumed that the system dynamics are described by

$$\dot{\underline{x}} = \underline{f}\,(t, \underline{x}, \underline{u}, \underline{\eta}) \quad ; \quad \underline{x}(t_1) = \underline{x}_1 \tag{1.1}$$

where $\underline{x}$ is the n dimensional state vector representing the process variables and $\underline{\eta}(t)$ is an m dimensional vector of uncertain parameters. The problem is to choose an r dimensional control function $\underline{u}$, assuming that $\eta = \eta_n$, such that over the time interval (t_1, t_2) the cost functional

$$J = \int_{t_1}^{t_2} L(t, \underline{x}, \underline{u})\, dt \tag{1.2}$$

is minimized subject to state and control constraints of the form

$$\underline{\phi}\,(t, \underline{x}, \underline{u}) < 0 \ . \tag{1.3}$$

A terminal region may or may not be specified. The solution of this problem yields a time varying control, $\underline{u} = \underline{u}_n(t)$, such that, when applied to (1.1) with the parameter vector at its nominal value, $\underline{\eta} = \underline{\eta}_n$, the optimal trajectory, $\underline{x}_n(t)$, is obtained. There are many quantities related to the optimization problem, such as the cost functional, constraint boundaries and terminal manifold, which are effected by parameter variations. However, all of these are related to the basic objective of generating the desired system output $\underline{x}_n(t)$.

The problem examined in this dissertation is the realization of $\underline{x}_n(t)$ when the actual system deviates from the design system through changes in the parameter vector $\underline{\eta}$. The trajectory sensitivity problem is thus defined to minimize or reduce variations in the system output $\Delta\underline{x} = \underline{x} - \underline{x}_n$, caused by variations or uncertainties in the modeling parameters, $\Delta\eta = \eta - \eta_n$.

1.2 Previous Work

During the past few years there has been much interest in the sensitivity of control systems, particularly those which are optimal in some sense. The following paragraphs contain a short description of the major results which have been obtained thus far.

Classical Sensitivity Techniques

Standard methods of solving the trajectory sensitivity problem for linear systems are described in reference [1] through [4]. The methods basically employ feedback as a second degree of freedom to reduce output errors as shown in Figure 1.1

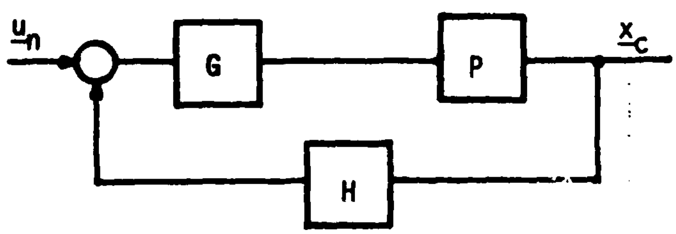

Figure 1.1: Closed Loop Control System

The first degree of freedom determines the nominal plant transfer function or operator P such that

$$\underline{x}_n(t) = P(\underline{u}_n(t)) \quad . \tag{1.4}$$

This is obtained, for example, from the solution to an optimal control problem. The feedback operator H is then determined to minimize or reduce sensitivity to plant parameter variations δP with G determined such that the overall (closed loop) transfer function is nominally equivalent, i.e., $\underline{x}_c = \underline{x}_n$ for $\delta P = 0$. The closed loop transfer function is thus

$$\underline{x}_c = [\ I - P\ G\ H\ ]^{-1}\ PG\ \underline{u}_n \quad . \tag{1.5}$$

It is therefore seen that, through the use of feedback, the structure of the original transfer function or operator is changed. In effect, two separate transfer functions P and D could be respectively determined to realize the original design objectives and to reduce trajectory sensitivity. The synthesis problem would then be to compute from these the operators G and H. To date, no practical design techniques had been determined to generate a physically realizable feedback operator, H, which minimizes the trajectory sensitivity.

Sensitivity Operators

The original definition of the sensitivity operator (references [1] - [4]) for single input, single output systems, relative to a scalar parameter η, is given by

$$S^P_\eta = {}^{\delta P/P}\!\Big/_{\delta\eta/\eta} \tag{1.6}$$

where P is the nominal plant transfer function (1.4). The operator relates normalized changes in the transfer function to parameter variations. Sensitivity is reduced by causing S^P_η to be as small as possible.

A generalization of this to multivariable systems described by linear operators was given first in [5] and later in [6] through [9]. It is assumed that the actual plant operator is given by

$$P_a = P + \delta P \tag{1.7}$$

where δP is an additive parameter variation. Referring to Figure 1.2 the closed loop output error, $\delta\underline{x}_c = \underline{x}_c - \underline{x}_n$, is related to the open loop error, $\delta\underline{x}_o = P_a\,\underline{u}_n - \underline{x}_n$, as follows

$$\delta\underline{x}_c = N \;\; \delta\underline{x}_o \tag{1.8}$$

where

$$N = [\ I - P_a\ G\ H\ ]^{-1}$$

but reduces to

$$N = [\ I + P\ H\ ]$$

when δP is small. The compensator is then

$$G = [\ I + H\ P\ ]^{-1}$$

for nominal equivalence. It was shown in [5] - [9] that a sufficient condition for sensitivity reduction defined by

$$\|\ \delta\underline{x}_c\ \| \leq \|\ \delta\underline{x}_o\ \|$$

was that the operator N be such that

$$[\ I - N^{*}\ N\ ] \geq 0 \tag{1.9}$$

when N^{*} is the adjoint of N, or that N be a contraction. Some stability conditions have been obtained for linear feedback gains in [8] and [9] such that (1.9) is satisfied. In general, (1.9) is difficult to use as a design tool.

<u>Closed Loop Optimal Systems</u>

Consider a general optimal control problem such as that given

by equations (1.1) through (1.3). One method of generating a feedback solution is to seek the optimal control as an explicit function of the initial state, $\underline{x}(t_1)$. Then t_1 can be treated as the present time with $\underline{x}(t_1)$ the present state. This has been done in references [10] and [11] for linear systems with quadratic cost functionals. The feedback gain matrix was shown to satisfy a matrix Riccati equation.

Since feedback does not necessarily imply sensitivity reduction or even stability, it is of interest to examine the sensitivity characteristics of optimal systems. The steady state regulator was considered in references [12] and [13] with nonlinear optimal systems being investigated in [14]. It was shown in [14] that for relatively smooth nonlinear systems

$$\int_{t_1}^{t'} \delta\underline{x}_c^T Z \, \delta\underline{x}_c \, dt \quad < \quad \int_{t_1}^{t'} \delta\underline{x}_o^T Z \, \delta\underline{x}_o \, dt \tag{1.10}$$

where Z is positive definite and t' is any point within the optimization interval. For linear quadratic problems

$$Z = K^T R K$$

where K is the feedback gain and R is the control cost weighting matrix. In addition, it was shown in [12] and [13] that for steady state linear quadratic problems (1.10) is equivalent to (1.9) and the classical return difference function [2] is greater than one.

In most cases, the optimal control cannot be obtained as an explicit function of the initial state and the above results then do not apply. Also (1.10) only states that a sensitivity reduction occurs and gives no indication of the actual amount of the reduction.

Sensitivity Functions

One method of reducing sensitivity in optimal control systems is to include sensitivity terms in the original cost functional. If the resulting control is implemented at a function of time (open loop),

then a tradeoff can be made between the original design and sensitivity objectives. This was done in reference [15] however, in order to achieve a significant sensitivity reduction, the original design objectives had to be considerably relaxed. This method also does not yield nominally equivalent solutions, i.e. the original (no sensitivity constraints) optimal control will not result when parameters are at their nominal values.

For the linear regulator problem, attempts were made to generate feedback controls with sensitivity terms in the cost. In order to do this, higher order sensitivity terms were neglected in references [16] and [17]. This approximation was avoided in [18] by treating control sensitivity terms as additional control functions. The resulting feedback control is linear in the state and the first order sensitivity vector, which is of the same dimension as the state. Therefore, the implementation of this requires the generation of the sensitivity terms by a dynamical system which is approaching the complexity of an optimal filter. *In addition*, since the regulator solution alone reduces sensitivity by (1.10), it is possible that the sensitivity reduction resulting from the augmented system could be simply obtained by adjustment of the terms in the original cost function.

Model Reference Adaptive Control

A particular solution to a control problem results in a nominal input to a plant with a specific transfer function. For this input, a model of the plant can be constructed which gives the desired output. The model reference technique [19] compares the actual output with the desired output and then adjusts certain control parameters such that a measure of the output error is minimized. Gradient and steepest descent procedures are used to determine the control parameters. This method has inherent stability problems in addition to being difficult to implement due to its complexity.

Linear Estimation and Control

Assume that for the system defined by equation (1.1), an optimal control function $\underline{u}_{\eta}(t)$ has been determined with $\underline{\eta} = \underline{\eta}_{\eta}$. Using the resulting optimal trajectory, $\underline{x}_{\eta}(t)$, as the nominal, the perturbation equation of (1.1) is

$$\Delta\dot{\underline{x}} = \frac{\partial f}{\partial x} \Delta\underline{x} + \frac{\partial f}{\partial u} \Delta\underline{u} + \frac{\partial f}{\partial \eta} \Delta\underline{\eta} \tag{1.11}$$

where all partials are evaluated along the nominal and the Δ quantities represent off nominal errors. Let some linear function of the state error

$$\Delta\underline{y} = M \Delta\underline{x} + \underline{d} \tag{1.12}$$

represent the measurements where $\underline{d}$ is Gaussian noise with zero mean and known variance. Then, assuming disturbances of known statistics on (1.11) and the linearized dynamics for $\Delta\underline{\eta}$, the Kalman filtering techniques described in [20] - [22] can be employed to estimate $\Delta\underline{x}$ and $\Delta\underline{\eta}$. This will require a dynamical system of dimension (n+m).

The control error, $\Delta\underline{u}$, can be determined such that a quadratic functional in $\Delta\underline{x}$ and $\Delta\underline{u}$ is minimized. In this case, references [10] and [11] indicate that $\Delta\underline{u}$ will be a linear function of $\Delta\underline{x}$ and $\Delta\underline{\eta}$. Because of the separation theorem ([23] and [24]) for linear, quadratic problems, the optimal filter and control can be combined to yield a dynamic feedback control, $\Delta\underline{u} = H(\Delta\underline{y})$. This is a good solution to the first order sensitivity problem except for its complexity and corresponding implementation difficulties. One other problem is the choice of state noise on the parameter error equation to keep the filter from becoming over confident (see reference [19]).

Feedback Operators and Controls

One technique of generating feedback controls from a given optimal control is to break it up into functions of time and state, e.g.

$$\underline{u}(t,\underline{x}) = \underline{u}_n(t) + \underline{h}(t,\underline{x}) \quad . \qquad (1.13)$$

where $\underline{h}(t,\underline{x}_n) = 0$ along the optimal trajectory. This type of control has been termed "partially closed loop" since $\underline{u}(t,\underline{x})$ is not an explicit function of the initial state. It was used in reference [25] to combat the singularity problem in final value control systems. Controls of the type (1.13) were also employed in reference [26] to reformulate the combined optimal control and sensitivity problem discussed in the section on sensitivity functions. The formulation basically considered $\underline{u}_n(t)$ and $[\ \partial h/\partial x\ ]$ as control functions to be determined by minimization of a coupled state-sensitivity cost functional. Reference [26] did not carry the problem further than the formulation.

Some interesting results have been obtained by Porter in [27] where the sensitivity problem was formulated using Functional Analysis. It can be seen from (1.8) that for small parameter variations, and when the compensator is $H = \lambda P^{-1}$ with λ a scalar, the sensitivity operator becomes

$$N = (\ 1 - \lambda\ )\ I \quad .$$

Thus as $\lambda \rightarrow 1$ the closed loop sensitivity approaches zero, however, the forward loop compensator gain (G) then becomes unbounded. The major drawback is that when P represents a differential equation, H is a differential (unbounded) operator and thus is difficult to implement. Also proposed in [27] is the following problem. Determine the operator H such that

$$J = [\ \| R\ \delta\underline{x}_c \|^2 + \| QH\ \delta\underline{x}_o \|^2\ ]$$

is a minimum where Q and R represent appropriate design matrices and the δx terms are as defined for (1.8). Using Hilbert space techniques, the solution is

$$H = \alpha[I + \alpha P^* P]^{-1} P^*$$

$$G = [I + \alpha P^* P]$$

where α is determined from R. When P represents a linear differential equation, the control error is given by

$$\underline{\Delta u} = -\alpha \int_{t}^{t_f} \phi^T (t_f, s)\, \underline{\Delta x}(s)\, ds$$

where $\phi(\cdot)$ is the transition matrix of P and t_f is the final time. The feedback control thus requires knowledge of future values of the state and is therefore unrealizable. It should be noted that this result can also be obtained using methods of variational calculus.

1.3 Scope of the Dissertation

The principal objective of this dissertation is to develop a new formulation of the trajectory sensitivity problem which is applicable to general nonlinear systems. As in classical sensitivity methods, the use of feedback as a second degree of freedom plays a large role in the theoretical development. To this end, the original system (1.1) with the control and parameter vectors at their design values ($\underline{u} = \underline{u}_n$, $\underline{\eta} = \underline{\eta}_n$) is considered as the nominal. A closed loop system function $\underline{g}(t, \underline{x}, \underline{\eta})$ is then sought such that the solution to

$$\dot{\underline{x}} = \underline{g}(t, \underline{x}, \underline{\eta}) \quad ; \quad \underline{x}(t_1) = \underline{x}_1 \tag{1.14}$$

remains close to the desired output of (1.1), $\underline{x}_n(t)$, when $\underline{\eta}$ differs from $\underline{\eta}_n$. The system equations (1.1) and (1.14) are related through a nominal equivalence condition, i.e.

$$\underline{g}(t, \underline{x}, \underline{\eta}_n) = \underline{f}(t, \underline{x}, \underline{u}_n, \underline{\eta}_n) .$$

An additional relationship can be obtained if the functions $g(\cdot)$ are generated by applying feedback controls to (1.1) as follows

$$\underline{g}(t, \underline{x}, \underline{\eta}) = \underline{f}(t, \underline{x}, \underline{u}(t,\underline{x}), \underline{\eta}) .$$

This is one possible solution to the synthesis problem.

In Chapter 2 the problem of determining the system function $\underline{g}(\cdot)$ which exhibits minimum output sensitivity to parameter errors is formulated as a direction field problem in the calculus of variations. Then, from a general class of measurable, quasiconvex, nominally equivalent system functions, necessary conditions are obtained for extremality. The extremal function effectively results from a trade-off between reduced output error and increased deviation from the nominal (design) system. These results are then shown to be applicable when the class of system functions is generated from feedback controls applied to the original system.

To alleviate the synthesis problem, further specialization is made in Chapter 2 to the class of linear, time varying gain feedback controls, small plant parameter variations and quadratic cost functions. The necessary conditions thus obtained explicitly determine the feedback gain matrix which minimizes the mean square and final value first order trajectory sensitivity, independent of the parameter errors. In Chapter 4, sufficiency conditions are determined using some new results on conjugate points derived in Appendix B. It is shown that the minimum sensitive (MS) gain function is determined, when it exists, from solutions to linear differential equations. Existence problems are removed through the addition of cost penality terms and linear approximations to the modified problem are then proposed. The theory developed in Chapter 4 gives, for the first time, a practical method of designing linear feedback compensators which minimize trajectory sensitivity.

The remaining chapters correlate presently known sensitivity reduction techniques to those of Chapter 4. Some new sensitivity relationships are derived in Chapter 3 for model following, regulating and stabilizing controls. A comparison example is presented in Chapter 5 which shows the superior sensitivity characteristics of the MS gain function relative to the controls discussed in Chapter 3. In addition, relationships between the MS compensator and classical sensitivity techniques for linear systems are derived in Appendix A.

It is shown that the feedback compensator of Chapter 4 is similar in structure to classical input compensators and that the closed loop system error, which is employed in Chapter 2 to limit the amount of applied feedback, can be corresponded to the transfer function for measurement noise. The latter is the limiting factor for classical sensitivity reduction techniques.

Least square parameter estimators are compared to MS controllers in Appendix C. It is shown that both have similar structures when the number of parameters equals the state dimension. The MS controller, however, is applicable to a much larger class of problems than the estimator and also is computationally simpler to determine.

Chapter 2

Direction Field Formulation of the Trajectory Sensitivity Problem

2.1 Introduction and Problem Statement

The solution of an optimal control problem over the time interval $[t_1, t_2]$ can be described by the following differential equation

$$\dot{\underline{x}}_n = \underline{f}(t, \underline{x}_n, \underline{u}_n, \underline{\eta}_n) \quad ; \quad \underline{x}_n(t_1) = \underline{x}_{n1} \tag{2.1}$$

where $\underline{x}$, $\underline{u}$ and $\underline{\eta}$ are of dimensions n, r and m respectively. The function $\underline{f}(\cdot)$ is assumed to be locally integrable in t and of class C^1 (continuous partial derivatives) in $\underline{x}$, $\underline{u}$ and $\underline{\eta}$. It is also assumed that the parameter vector $\underline{\eta}(t)$ evolves according to

$$\dot{\underline{\eta}} = \underline{\ell}(t, \underline{\eta}) \quad ; \quad \underline{\eta}(t_1) = \underline{\eta}_1 \tag{2.2}$$

where $\underline{\ell}(\cdot)$ is a fixed function that is locally integrable in t and C^1 in $\underline{\eta}$. The solution of (2.1) is given by

$$\underline{x}_n(t, \underline{\eta}_n) = \underline{x}_n(t) \tag{2.3}$$

which is the desired optimal trajectory. Assuming that all initial condition errors are accounted for in $\underline{u}_n(t)$, variations in the modeling parameters, $\Delta\underline{\eta} = \underline{\eta}(t) - \underline{\eta}_n(t)$, result in the open loop system

$$\dot{\underline{x}}_o = \underline{f}(t, \underline{x}_o, \underline{u}_n, \underline{\eta}) \quad ; \quad \underline{x}_o(t_1) = \underline{x}_{n1} \quad . \tag{2.4}$$

The corresponding open loop trajectory is thus

$$\underline{x}_o(t, \underline{\eta}) = \underline{x}_o(t) \tag{2.5}$$

which is absolutely continuous in t and C^1 in $\underline{\eta}$.

Feedback can be employed as a second degree of freedom by defining the closed loop system as

$$\dot{\underline{x}} = \underline{g}(t, \underline{x}, \underline{\eta}) \quad ; \quad \underline{x}(t_1) = \underline{x}_{n1} \tag{2.6}$$

where $\underline{g}(\cdot)$ is a member of a certain class of functions G to be specified in the next section. Each $\underline{g}(\cdot) \in G$ satisfies the nominal equivalence condition

$$\underline{g}(t, \underline{x}, \underline{\eta}_n) = \underline{f}(t, \underline{x}, \underline{u}_n, \underline{\eta}_n) \ . \tag{2.7}$$

Therefore (2.6) defines a class of direction fields about $\underline{f}(\cdot)$ which is parameterized by $\underline{\eta}(t)$. In order to solve the trajectory sensitivity problem defined in Chapter 1, a function $\underline{g}(\cdot) \in G$ must be chosen such that the solution of (2.6), $\underline{x}(t, \underline{\eta})$, minimizes some function of the error

$$\Delta\underline{x}(t, \underline{\eta}) = \underline{x}(t, \underline{\eta}) - \underline{x}_n(t) \tag{2.8}$$

over the original optimization interval $[t_1, t_2]$. For practical reasons such as system noise and original design constraints, $g(\cdot)$ must also be chosen to limit some function of the system error

$$\Delta\underline{g}(t, \underline{x}, \underline{\eta}) = \underline{g}(t, \underline{x}, \underline{\eta}) - \underline{f}(t, \underline{x}, \underline{u}_n, \underline{\eta}) \ . \tag{2.9}$$

This represents the deviation of the closed loop system from the open loop system. The trajectory and system errors, (2.8) and (2.9) can be combined into a general cost function

$$J(t_2, \Delta\underline{g}) = \psi(t_2, \Delta\underline{x}) + \int_{t_1}^{t_2} L(t, \Delta\underline{x}, \Delta\underline{g})\, dt$$

or equivalently

$$J = \int_{t_1}^{t_2} I(t, \underline{x}, \underline{\eta}, \underline{g}(\cdot)\ )\, dt \tag{2.10}$$

where $I(\cdot)$ is assumed to be locally integrable in $t \; \forall \; \underline{g}(\cdot) \in G$ and C^1 WRT $\underline{x}$, $\underline{\eta}$ and $\underline{g}(\cdot)$. The above cost effectively trades off trajectory sensitivity for closed loop system error. The problem is thus to choose $\underline{g}(\cdot) \in G$ such that (2.10) is minimized subject to (2.6).

2.2 Necessary Conditions for Minimum Sensitive Closed Loop System Functions

In the following paragraphs, the class of admissible functions G will be defined along with a general concept of an extremal. Then necessary conditions will be derived for the closed loop system function which minimizes trajectory sensitivity.

Admissible Class of Functions

Let G be an n dimensional family of functions $\underline{g}(t, \underline{x}, \underline{\eta})$ where

$t \in T$ - a bounded interval
$\underline{x} \in R_x$ - an open subset of R^n
$\underline{\eta} \in R_\eta$ - an open subset of R^m

It is assumed that the following hold for each $\underline{g}(\cdot) \in G$.

a) Each $\underline{g}(t, \underline{x}, \underline{\eta})$ is measurable in $t \in T$ for each $\underline{x} \in R_x$, $\underline{\eta} \in R_\eta$.

b) The functions $\underline{g}(\cdot)$ are in class C^1WRT $\underline{x}$ and $\underline{\eta}$.

c) To each $\underline{g}(\cdot) \in G$ and compact subset $S \subset R_x \times R_\eta$, $\exists$ an integrable function $m(t)$ on T st $\forall \; \underline{x}, \underline{\eta} \in S$

$$\| \underline{g}(t, \underline{x}, \underline{\eta}) \| < m(t)$$
$$\| \underline{g}_x(t, \underline{x}, \underline{\eta}) \| < m(t)$$
$$\| \underline{g}_\eta(t, \underline{x}, \underline{\eta}) \| < m(t)$$

where the norms are the standard vector and matrix norms in Euclidean space.

d) Define $P^b = \{ \underline{\alpha} \in R^b : \alpha_i > 0, \sum_{i=1}^{b} \alpha_i = 1 \}$

$$[G] = \text{convex hull of } G$$

$$[G] = \{ \underline{g}^h : \underline{g}^h(t, \underline{x}, \underline{\eta}) = \sum_{i=1}^{b} \alpha_i \, \underline{g}^i (t, \underline{x}, \underline{\eta}) \, , \quad \underline{\alpha} \in P^b \, , \, g^i \in G \}$$

Then to each set $\{ g^i \in G \}$ $i = 1,\ldots,b$, $\underline{\alpha} \in P^b$, and $\varepsilon > 0$ $\exists\, g^\alpha \in G$ st the function

$$\underline{h}(t, \underline{x}, \underline{\eta}; \alpha) = \sum_{i=1}^{b} \alpha_i \, \underline{g}^i(t, \underline{x}, \underline{\eta}) - \underline{g}^\alpha(t, \underline{x}, \underline{\eta})$$

satisfies the following conditions

d_1) $\exists$ an integrable function $\bar{m}(t)$st

$$\| \underline{h}(t, \underline{x}, \underline{\eta}; \alpha) \| \leq \bar{m}(t)$$

$$\| \underline{h}_x(t, \underline{x}, \underline{\eta}; \alpha) \| \leq \bar{m}(t)$$

$$\| \underline{h}_\eta(t, \underline{x}, \underline{\eta}; \alpha) \| \leq \bar{m}(t)$$

$$\forall\, \underline{x}, \underline{\eta} \in S \quad ; \quad t \in T \; ; \quad \underline{\alpha} \in P^b$$

d_2) $\| \int_{t'_1}^{t'_2} \underline{h}(t, \underline{x}, \underline{\eta}; \alpha) \, dt \| < \varepsilon$

$$\forall\, \underline{x}, \underline{\eta} \in S \quad ; \quad \underline{\alpha} \in P^b \quad ; \quad t'_1, t'_2 \in T$$

d_3) For each sequence $\{ \underline{\alpha}^j \} \in P^b$ st $\alpha^j \rightarrow \bar{\underline{\alpha}} \in P^b$ as $j \rightarrow \infty$ and each $\delta > 0$,

$$\mu\{t: \| \underline{h}(t, \underline{x}, \underline{\eta}; \underline{\alpha}^j) - \underline{h}(t, \underline{x}, \underline{\eta}, \bar{\underline{\alpha}}) \| > \delta\} \rightarrow 0$$

as $j \rightarrow \infty$ $\forall\, \underline{x}, \underline{\eta} \in S$ where μ is a Lebesgue measure on T.

e) Each $\underline{g}(\cdot)$ is nominally equivalent WRT $\underline{\eta}$, i.e.

$$\underline{g}(t, \underline{x}, \underline{\eta}_n) = \underline{f}(t, \underline{x}, \underline{u}_n, \underline{\eta}_n) \, .$$

Assumptions a), b) and c) are basically minimal requirements for the differential equation (2.6) to have a unique solution (reference [30]).

The quasiconvexity assumption d) is employed to assure that enough functions are in G for the extremal problem to be meaningful.

Definition of Extremality

For some element $\underline{g}(\cdot) \in G$, the system differential equations are

$$\dot{\underline{x}} = \underline{g}(t, \underline{x}, \underline{\eta})$$
$$\dot{\underline{\eta}} = \underline{\ell}(t, \underline{\eta}) \tag{2.11}$$

where $\underline{\ell}(\cdot)$ is defined by (2.2) and satisfies the same conditions as each $\underline{g}(\cdot) \in G$ WRT t and $\underline{\eta}$, although it is a fixed (or given) function. The solution to (2.11) is

$$\underline{x}(t, \underline{\eta}) \quad , \quad \underline{\eta}(t) \quad ; \quad t \in T$$

where both are absolutely continuous a.e. in t and C^1 WRT $\underline{\eta}$ with boundary conditions either implied or explicitly stated

$$\underline{x}(t_1) = \underline{x}_1 \quad ; \quad \underline{x}(t_2) = \underline{x}_2$$
$$\underline{\eta}(t_1) = \underline{\eta}_1 \quad ; \quad \underline{\eta}(t_2) = \underline{\eta}_2 \quad .$$

Let

$$q_{x\eta} = (t_1, t_2, \underline{x}_1, \underline{\eta}_1, \underline{x}_2, \underline{\eta}_2) \in R^{2(n+m+1)} \tag{2.12}$$

Define a set $Q \subset R^{2(n+m+1)}$ as that containing all points $q_{x\eta}$ corresponding to all solutions of (2.11) for all $\underline{g}(\cdot) \in G$. An extremal will be defined in terms of a given differentiable manifold $N \subset R^{2(n+m+1)}$ with boundary M as follows.

Definition: The solution $\bar{\underline{x}}$, $\bar{\underline{\eta}}$ of (2.11) for some $\bar{\underline{g}}(\cdot) \in G$ is a G - N extremal if $q_{\bar{x}\bar{\eta}} \in M$ and if $\exists$ an open neighborhood $U \ni q_{\bar{x}\bar{\eta}}$ st $U \cap N \cap Q \subset M$.

The above implies that at $q_{\bar{x}\bar{\eta}}$ the sets N and Q are separated in some sense.

The manifold N can be defined in terms of the sensitivity problem by considering the cost function J given by (2.10). Adjoin to the state vector $\underline{x}(t)$ the quantity $x_{(n+1)}(t)$ defined from

$$\dot{x}_{(n+1)}(t) = I\ (t, \underline{x}, \underline{\eta}, \underline{g}(\cdot)\ ) \quad ; \quad x_{(n+1)}(t_1) = 0 \qquad (2.13)$$

and extend Q to include all points

$$q_{x\eta} = (t_1,\ t_2,\ \underline{x}_1,\ x_{(n+1)}(t_1),\ \underline{\eta}_1,\ \underline{x}_2,\ x_{(n+1)}(t_2), \underline{\eta}_2) \in R^{2(n+m+2)} .$$

Using the fact that the optimization interval is fixed along with the initial state and parameter vectors, define the differentiable manifold $N \subset R^{2(n+m+2)}$ as the set of points

$$(\nu_1,\ \nu_2,\ \underline{\xi}_1,\ \zeta_1,\ \underline{\gamma}_1,\ \underline{\xi}_2,\ \zeta_2,\ \underline{\gamma}_2)$$

given by

$$\begin{aligned} &\nu_1 = t_1 \quad , \quad \nu_2 = t_2 \\ &\underline{\xi}_1 = \underline{x}_1 \quad , \quad \zeta_1 = 0\ , \ \underline{\gamma}_1 = \underline{\eta}_1 \end{aligned} \qquad (2.14)$$

which are fixed quantities and by

$$\begin{aligned} &\underline{\xi}_2 \in R_x \\ &\underline{\gamma}_2 \in \underline{R}_\eta \\ &\zeta_2 < \bar{x}_{(n+1)}(t_2) \end{aligned} \qquad (2.15)$$

where $\bar{x}_{(n+1)}(t_2)$ is the minimum of J. Thus N is a subset of at most dimension (n+m+1). The boundary M is given by all relations defining N except that the one involving ζ_2 is replaced by

$$\zeta_2 = \bar{x}_{(n+1)}(t_2) \ . \qquad (2.16)$$

Necessary Conditions

Let G′ be an (n+m+1) dimensional family of functions given by

$$\underline{g}'(t, \underline{z}) = [\ \underline{g}^T(t, \underline{x}, \underline{\eta})\ ,\ I(t, \underline{x}, \underline{\eta}, \underline{g}(\cdot)\ ),\ \underline{\ell}^T(t, \underline{\eta})\ ]^T$$

where

$$\underline{z}(t) = [\ \underline{x}^T(t, \underline{\eta})\ ,\ x_{(n+1)}(t),\ \underline{\eta}^T(t)\ ]^T$$

with $\underline{g}(\cdot) \in G$ and $\underline{\ell}(\cdot)$ a function previously defined satisfying conditions a) through c). The cost integrand $I(\cdot)$ is assumed to satisfy a) through d) $\forall\ \underline{g}(\cdot) \in G$. Thus each $g'(\cdot) \in G'$ is an admissible function. The quasiconvexity condition carries over for $\ell(\cdot)$ because it is fixed. Define the Hamiltonian as

$$H\ (t, \underline{z}, \underline{\psi}) = \underline{\psi}^T(t)\ \underline{g}'\ \ (t, \underline{z})$$

where $\underline{\psi}(t)$ is an absolutely continuous (n+m+1) dimensional vector function. Note that $H(\cdot)$ is completely determined by the choice of $\underline{g}'(\cdot) \in G'$.

As previously stated, the sensitivity problem is to choose $\underline{g}(\cdot) \in G$ such that (2.10) is minimized subject to (2.6). The following theorem gives necessary conditions for the minimizing closed loop system function.

Theorem 2.2: Let the function $\bar{g}(\cdot) \in G$ and corresponding solution $\underline{\bar{x}}(t, \underline{\bar{\eta}})$ of

$$\dot{\underline{\bar{x}}} = \underline{\bar{g}}\ (t, \underline{\bar{x}}, \underline{\bar{\eta}})\ ;\ \underline{\bar{x}}(t_1) = \underline{x}_{n1} \tag{2.17}$$

with $\underline{\bar{\eta}}(t)$ given by (2.2) minimize the sensitivity cost (2.10) over the interval $[t_1, t_2]$. Then $\exists$ a nontrivial absolutely continuous vector $\underline{\psi}(t)$ on $[t_1, t_2]$ st with $\underline{\bar{z}} = [\ \underline{\bar{x}}^T,\ \bar{x}_{n+1},\ \underline{\bar{\eta}}^T]^T$ the following holds

$$\dot{\underline{\bar{z}}} = \partial\bar{H}\ (t, \underline{\bar{z}}, \underline{\psi})\ /\ \partial\underline{\psi}\ = \underline{\bar{g}}'\ (t, \underline{\bar{z}}\ )$$

$$\dot{\underline{\psi}} = -\ \partial\bar{H}\ (t, \underline{\bar{z}}, \underline{\psi})\ /\ \partial\underline{z} = -\ \underline{\psi}^T(t)\ \bar{g}'_z(t, \underline{\bar{z}})$$

where $\bar{g}(t, \bar{z}) = [\, \bar{g}^T(t, \bar{x}, \bar{\eta}),\ I(t, \bar{x}, \bar{\eta}, \bar{g}(\cdot)\,),\ \underline{\ell}^T(t, \bar{\eta})\,]^T$

and $\bar{H}(t, \bar{z}, \underline{\psi}) = \underline{\psi}^T(t)\, \bar{g}(t, \bar{z})$.

In addition $\forall\ g'(\cdot) \in G'$

$$\int_{t_1}^{t_2} \bar{H}(t, \bar{z}, \underline{\psi})\, dt \geq \int_{t_1}^{t_2} H(t, \bar{z}, \underline{\psi})\, dt \ .$$

If $\bar{g}(t, \bar{z})$ is continuous in t at t_1 and t_2, then the transversality condition is that

$$[\, \bar{H}(t_1, \bar{z}, \underline{\psi}),\ -\bar{H}(t_2, \bar{z}, \underline{\psi}),\ -\underline{\psi}^T(t_1),\ \underline{\psi}^T(t_2)\,]$$

be orthogonal to the boundary M at the point $q_{\bar{z}}$.

Proof: The above conclusions are similar to those of Theorem 2.1 in reference [29], the statement and proof of which are given in sections 2 and 3 of the reference. The theorem differs in that [29] assumes that for a particular $\bar{g}(\cdot) \in G'$, the solution $\bar{z}(t)$ of

$$\dot{\bar{z}} = \bar{g}(t, \bar{z})$$

on the inverval $[\, t_1, t_2\,]$ is a G' - N extremal. It therefore remains to show that the hypotheses of this theorem imply a G' - N extremal. To this end, let the manifold N with boundary M be as defined in the previous section. Since the cost (2.10) is a minimum and is represented by $\bar{x}_{(n+1)}(t_2)$, any point in Q will be such that

$$x_{(n+1)}(t_2) \geq \bar{x}_{(n+1)}(t_2) \ .$$

Thus the intersection of any open set about $q_{\bar{z}}$ with Q and N will result in $\zeta_2 = x_{(n+1)}(t_2) = \bar{x}_{(n+1)}(t_2)$ which is by definition in M. The solution $\bar{z}(t)$ is therefore a G' - N extremal which completes the proof.

It should be noted that the nominal equivalence condition, e), is not explicitly required to obtain the above necessary conditions. It is a synthesis requirement which will become more explicit in later sections of this chapter where the class of system functions is generated by nominally equivalent feedback controls.

If the cost function (2.10) is sufficiently convex, certain bounds can be obtained on the minimizing closed loop system (2.17). For example, when no restriction is placed on the system error $\Delta\underline{g}(t, \underline{x}, \underline{\eta})$ given by (2.9), then the output error and equivalently the cost J can be made zero over $[t_1, t_2]$ by choosing

$$\underline{\bar{g}}\,(t, \underline{x}, \underline{\eta}) = \underline{f}(t, \underline{x}, \underline{u}_n, \underline{\eta}_n) \quad .$$

This not only requires knowledge of the parameter $\underline{\eta}(t)$ but controllability as well. If no restrictions are placed on the output error, then zero cost will result from

$$\underline{\bar{g}}(t, \underline{x}, \underline{\eta}) = f(t, \underline{x}, \underline{u}_n, \underline{\eta})$$

which is the open loop system. It is therefore seen that particular choices of the cost functional (2.10) can not only alter the closed loop system function but also the synthesis problem.

2.3 Generation of System Functions Using Feedback Controls

The standard technique of producing closed loop systems is by applying feedback controls, $\underline{u}(t, \underline{x})$, to the original system (2.1) in place of the open loop control, $\underline{u}_n(t)$. Thus (2.6) becomes

$$\dot{\underline{x}} = \underline{f}(t, \underline{x}, \underline{u}(t, \underline{x}), \underline{\eta}) \quad ; \quad \underline{x}(t_1) = \underline{x}_{n1} \quad . \tag{2.18}$$

If $\underline{u}(t, \underline{x})$ is nominally equivalent, i.e.

$$\underline{u}(t, \underline{x}_n) = \underline{u}_n(t) \tag{2.19}$$

then (2.1) will be realized when the parameters are at their nominal

values, $\underline{\eta}_n(t)$. In this section, the explicit problem of determining a feedback control law which minimizes (2.10) subject to (2.18) will be considered. Therefore, let G_u be an n dimensional family of functions defined by

$$G_u = \{ \underline{g}(t, \underline{x}, \underline{\eta}) : \underline{g}(t, \underline{x}, \underline{\eta}) = \underline{f}(t, \underline{x}, \underline{u}(t, \underline{x}), \underline{\eta}) ; \underline{u}(\cdot) \in U\} \tag{2.20}$$

where the set U of admissible controls is defined below and $\underline{f}(\cdot)$ is the original system function given by (2.1). If the class G_u can be shown to be admissible, which implies satisfaction of conditions a) through e) of section 2.2, then Theorem 2.2 can be applied to determine the minimizing control.

Let U be the set of all control functions

$$\underline{u}(t, \underline{x}) : T \times R_x \to Y$$

where T and R_x are defined in section 2.2 and Y is a fixed set in R^r. It is assumed that for each $\underline{u}(\cdot) \in U$ the following conditions hold,

m) Each $\underline{u}(t, \underline{x})$ is measurable in $t \in T$ for each $\underline{x} \in R_x$.

n) The functions $\underline{u}(\cdot)$ are in class C^1 WRT $\underline{x}$.

o) For every $\underline{u}(t, \underline{x}) \in U$, $\underline{f}(t, \underline{x}, \underline{u}(\cdot), \underline{\eta})$ measurable in t and C^1 WRT $\underline{x}$, $\underline{u}$, $\underline{\eta}$, and compact subset $S \subset R_x \times R_\eta$, $\exists$ an integrable function $\bar{m}(t)$ possibly depending on S, $\underline{f}(\cdot)$ and $\underline{u}(\cdot)$ st $\forall$ $\underline{x}, \underline{\eta} \in S$

$$\| \underline{f}(t, \underline{x}, \underline{u}(t, \underline{x}), \underline{\eta} \| < \bar{m}(t)$$

$$\| \underline{f}_x(t, \underline{x}, \underline{u}(t, \underline{x}), \underline{\eta}) \| < \bar{m}(t)$$

$$\| \underline{f}_u(t, \underline{x}, \underline{u}(t, \underline{x}), \underline{\eta}) \cdot \underline{u}_x (t, \underline{x}) \| < \bar{m}(t)$$

$$\| \underline{f}_\eta(t, \underline{x}, \underline{u}(t, \underline{x}), \underline{\eta}) \| < \bar{m}(t)$$

where $\| \cdot \|$ represents the Euclidean norm.

p) The control function

$$\underline{u}'(t, \underline{x}) = \begin{cases} \underline{u}^1(t, \underline{x}) & , \quad t \in T_1 \\ \underline{u}^2(t, \underline{x}) & , \quad t \notin T_1 \end{cases}$$

is in the class U where $\underline{u}^1(\cdot), \underline{u}^2(\cdot) \in U$ and $T_1 \subset T$.

q) The control functions are nominally equivalent, i.e.

$$\underline{u}(t, \underline{x}_n) = \underline{u}_n(t) \ .$$

Conditions m) through o) are included to assure unique solutions to the differential equation (2.18). Property p) guarantees that control perturbations such as those used in [32] are admissible.

The next Theorem extends the results of [29] for time varying controls to include feedback control functions.

Theorem 2.3: The class of system functions G_u defined by (2.20) using controls $\underline{u}(t, \underline{x}) \in U$ is admissible, i.e., the functions satisfy conditions a) through e) of section 2.2.

Proof: From the assumptions on the original system function given by (2.1) and conditions m) through o) defining U, it is seen that conditions a) through c) are satisfied for each $g(\cdot) \in G_u$. Also, by definition of nominal equivalence, q) implies e). It therefore remains to show that condition d) holds.

Let the functions $\{\underline{g}^i(\cdot) \in G_u\}$, $i = 1,\ldots,b$ be given along with $\underline{\alpha} \in P^b$, compact subset $S \subset R_x \times R_\eta$ and scalar $\varepsilon > 0$ where

$$\underline{g}^i(t, \underline{x}, \underline{\eta}) = \underline{f}(t, \underline{x}, \underline{u}^i(t, \underline{x}), \underline{\eta}) \ ; \ \underline{u}^i(\cdot) \in U \ . \qquad (2.21)$$

It is desired to show that $\exists$ a control $\underline{u}^\alpha(t, \underline{x}) \in U$ such that the function

$$\underline{h}(t, \underline{x}, \underline{\eta};\alpha) = \sum_{i=1}^{b} \alpha_i \, \underline{g}^i(t, \underline{x}, \underline{\eta}) - \underline{f}(t, \underline{x}, \underline{u}^\alpha(t, \underline{x}), \underline{\eta}) \qquad (2.22)$$

satisfies condition d), in particular the quasiconvexity condition d_2). In what follows the control will be shown to be of the form

$$\underline{u}^{\alpha}(t, \underline{x}) = \underline{u}^{i}(t, \underline{x}) \quad ; \quad t \in E_{ij}, \quad j = 1,\ldots,K$$

where the E_{ij} are disjoint, $\underline{x} \in S$ and $\bigcup_{i,j} E_{ij} \subset T$. The method of proof is similar to that of Lemma 4.1 in [29]; however, the structure is much less complex.

It was shown in [29] that there exist continuous functions on TxS arbitrarily close to $\underline{g}^{i}(t, \underline{x}, \underline{\eta})$ in the topology defined by d_2). Thus for the remainder of the proof, the $g^{i}(\cdot)$ will be assumed continuous in t for fixed $\underline{x}$ and $\underline{\eta}$. Partition $T = [\ t_1, t_2\ ]$ into disjoint subintervals I_j, $j = 1,\ldots,K$, and choose K sufficiently large st $\forall\ \underline{x}, \underline{\eta} \in S$ and $i = 1,\ldots,b$, the following holds

$$\| g^{i}(t_j, \underline{x}, \underline{\eta}) - \underline{g}^{i}(t', \underline{x}, \underline{\eta}) \| < {}^{\varepsilon}/2T \qquad (2.23)$$

$$\| \int_T \sum_{i=1}^{b} \alpha_i \, \underline{g}^{i}(t, \underline{x}, \underline{\eta}) \, dt - A(K) \| < {}^{\varepsilon}/2 \qquad (2.24)$$

where

$$A(K) = \sum_{j=1}^{K} \sum_{i=1}^{b} \alpha_i \, \underline{g}^{i}(t_j, \underline{x}, \underline{\eta}) \, I_j \qquad (2.25)$$

and t', $t_j \in I_j$. It is possible to do this by definition of the integral and of continuous functions and since S is compact. Now further divide each I_j into b disjoint subintervals

$$E_{ij} = \alpha_i \, I_j \qquad i = .,\ldots,b \quad . \qquad (2.26)$$

Note that

$$\bigcup_{i=1}^{b} E_{ij} = \sum_{i=1}^{b} \alpha_i \, I_j = I_j \quad .$$

Thus (2.25) becomes

$$A(K) = \sum_{j=1}^{K} \sum_{i=1}^{b} \underline{g}^{i}(t_j, \underline{x}, \underline{\eta}) \, E_{ij} \quad .$$

Define the control function as

$$\underline{u}^{\alpha}(t, \underline{x}) = \underline{u}^{i}(t, \underline{x}) \quad ; \quad t \in E_{ij} \tag{2.27}$$

and note that $\underline{u}^{\alpha}(\cdot) \in U$ by condition p). From (2.21)

$$\underline{g}^{\alpha}(t, \underline{x}, \underline{\eta}) = \underline{f}(t, \underline{x}, \underline{u}^{\alpha}(t, \underline{x}), \underline{\eta}) \quad . \tag{2.28}$$

Thus $\forall\ \underline{x}, \underline{\eta} \in S$

$$\| \int_T \underline{h}(t, \underline{x}, \underline{\eta}; \alpha)\, dt \|$$

$$= \| \int_T \sum_{i=1}^{b} \alpha_i\, \underline{g}^{i}(\cdot)\, dt - \int_T \underline{g}^{\alpha}(\cdot)\, dt \|$$

$$< \| \int_T \Sigma\, \alpha_i\, \underline{g}^{i}(\cdot)\, dt - A(K) \| + \| A(K) - \int_T \underline{g}^{\alpha}(\cdot)\, dt \|$$

$$< {}^{\epsilon}/2 + \| \sum_{j=1}^{K} \sum_{i=1}^{b} \underline{g}^{i}(t_j, \underline{x}, \underline{\eta})\, E_{ij} - \sum_{j=1}^{K} \sum_{i=1}^{b} \int_{E_{ij}} \underline{g}^{i}(t, \underline{x}, \underline{\eta})\, dt \|$$

$$< {}^{\epsilon}/2 + \sum_{j=1}^{K} \sum_{i=1}^{b} \| \int_{E_{ij}} (\underline{g}^{i}(t_j, \underline{x}, \underline{\eta}) - \underline{g}^{i}(t, \underline{x}, \underline{\eta}))\, dt \|$$

$$< {}^{\epsilon}/2 + {}^{\epsilon}/2T \sum_{j=1}^{K} \sum_{i=1}^{b} E_{ij}$$

$$< \epsilon$$

Since the above holds on a subset $[t_1', t_2'] \subset T$, condition d_2) is met. Also the fact that $\underline{h}(\cdot)$ in (2.22) is generated directly from $\underline{f}(\cdot)$ implies that d_1) and d_3) hold. Conditions a) through e) of section 2.2 are satisfied, and therefore the class G_u is admissible. This completes the proof.

Using the above theorem, it is possible to compare the problem

formulations of references [29] and [31]. Gittleman formulates an extremal problem in which the system functions must satisfy a condition similar to property p). The above theorem implies that functions which satisfy this property are quasiconvex. Therefore to this extent, Gittleman's formulation is included in that of Gamkrelidze.

2.4 Specialization to Small Parameter Variations, Linear Feedback Controls and Quadratic Cost

The problem of synthesizing from the class U defined in section 2.3 a feedback control such that the necessary conditions of Theorem 2.2 are satisfied is in general a difficult task. The difficulty occurs because the theorem gives no conditions for the structure of the minimizing control function, $\underline{u}(t, \underline{x})$. In what follows, further specialization of the sensitivity problem will be made such that the synthesis question is resolved by the problem formulation.

In addition to the conditions imposed on the admissible class of system functions G and on the control set U, the following assumptions are made.

s_1) The unknown parameter η is a scalar constant, i.e. $\ell(\cdot) = 0$ in (2.2). (Multiple parameter variations will be considered in Chapter 4.0).

s_2) The parameter error, $\Delta\eta = \eta - \eta_n$, is sufficiently small such that only first order terms are required to describe the system behavior relative to the nominal.

s_3) The closed loop system functions are generated by the family G_u defined in (2.20) with the control class U restricted to contain elements of the form

$$\underline{u}(t, \underline{x}) = \underline{u}_1(t) + K(t)\,\underline{x}(t) \tag{2.29}$$

where $\underline{u}_1(t)$ and the (rxn) matrix K(t) are essentially bounded functions ([33], [34]) with $\underline{u}_1(t)$ determined for nominal equivalence.

s_4) The cost functional J given by (2.10) is quadratic in

$\Delta\underline{x}(t, \eta)$ and $\Delta\underline{g}(t, \underline{x}, \eta)$, i.e.

$$J = \frac{1}{2}\, \Delta\underline{x}^T(t_2, \eta)\, D\, \Delta\underline{x}(t_2, \eta) + \frac{1}{2} \int_{t_1}^{t_2} [\, \Delta\underline{x}^T\, Q\, \Delta\underline{x} + \Delta\underline{g}^T\, W\, \Delta\underline{g}\,]\, dt \tag{2.30}$$

where D and Q are (nxn) positive semi-definite matrices and W is a positive definite (nxn) matrix. Also, Q is assumed to be integrable and W essentially bounded.

Using the above, explicit expressions can be obtained for the cost (2.30) and system differential equations (2.6) which involve only first order trajectory sensitivity terms.

The error expressions in (2.30) are evaluated as follows. Since the solution of (2.6) is C^1WRT η, assumptions s_1) and s_2) imply that

$$\begin{aligned} \Delta\underline{x}(t, \eta) &= \underline{x}(t, \eta) - \underline{x}_n(t, \eta_n) \\ &= \frac{\partial \underline{x}_n(t, \eta_n)}{\partial \eta}\, [\, \eta - \eta_n\,] \\ &\triangleq \underline{s}(t)\, \Delta\eta \end{aligned} \tag{2.31}$$

where $\underline{s}(t)$ represents the first order trajectory sensitivity. To determine $\Delta\underline{g}(t, \underline{x}, \eta)$, note that from s_1) and s_2)

$$\underline{f}(t, \underline{x}, \underline{u}_n, \eta) = \underline{f}(t, \underline{x}_n, \underline{u}_n, \eta_n) + [\, \frac{\partial f}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial \eta}\,]\, \Delta\eta$$

$$\underline{g}(t, \underline{x}, \eta) = \underline{g}(t, \underline{x}_n, \eta_n) + [\, \frac{\partial g}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial g}{\partial \eta}\,]\, \Delta\eta$$

where all partials exist because $\underline{f}(\cdot)$ and $\underline{g}(\cdot)$ are C^1WRT $\underline{x}$ and η. Thus

$$\Delta\underline{g}(t, \underline{x}, \eta) = \underline{g}(t, \underline{x}, \eta) - \underline{f}(t, \underline{x}, \underline{u}_n, \eta)$$

$$= \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x} \right] \frac{\partial x}{\partial \eta} \Delta\eta + \left[\frac{\partial g}{\partial \eta} - \frac{\partial f}{\partial \eta} \right] \Delta\eta \qquad (2.32)$$

since by assumption e) $\underline{g}(\cdot)$ is nominally equivalent. From s_3) and (2.20)

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u} \frac{\partial u}{\partial x}$$

$$\frac{\partial g}{\partial \eta} = \frac{\partial f}{\partial \eta} \quad .$$

Equation (2.32) thus becomes

$$\Delta\underline{g}(t, \underline{x}, \eta) = \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} \frac{\partial x}{\partial \eta} \Delta\eta$$

which yields, from (2.29) and (2.31)

$$\Delta\underline{g}(t, \underline{x}, \eta) = \frac{\partial f}{\partial u} (t, \underline{x}_n, \underline{u}_n, \eta_n) \cdot K(t) \, \underline{s}(t) \, \Delta\eta \quad . \qquad (2.33)$$

Using (2.31) and (2.33) in (2.30), an expression for the cost is

$$J = \frac{\Delta\eta^2}{2} \underline{s}^T(t_2) \, D \, \underline{s}(t_2) + \frac{\Delta\eta^2}{2} \int_{t_1}^{t_2} [\underline{s}^T Q \underline{s} + \underline{s}^T K^T \frac{\partial f}{\partial u}^T W \frac{\partial f}{\partial u} K \underline{s}] \, dt \ .$$

It is assumed that $\frac{\partial f}{\partial u} (t, \underline{x}_n, \underline{u}_n, \eta_n)$ is essentially bounded in t and of rank r. Then since the parameter error $\Delta\eta$ is uncontrollable and the matrix $[\frac{\partial f}{\partial u}^T W \frac{\partial f}{\partial u}]$ is positive semi-definite of rank r, it is sufficient to minimize over K(t)

$$J = \frac{1}{2} \underline{s}^T(t_2) \, D \, \underline{s}(t_2) + \frac{1}{2} \int_{t}^{t_2} [\underline{s}^T \, Q \, \underline{s} + \underline{s}^T K^T \, R \, K \, \underline{s}] \, dt \qquad (2.34)$$

where R is positive definite, essentially bounded and of rank r.

It remains to determine a differential equation for $\underline{s}(t)$. From (2.18) the system dynamics are given by

$$\dot{\underline{x}} = \underline{f}(t, \underline{x}, \underline{u}(t, \underline{x}), \eta); \; \underline{x}(t_1) = \underline{x}_{n1} \tag{2.35}$$

where $\underline{u}(\cdot) \in U$ is defined by (2.29). From s_1) and s_2), the first order expansion of (2.35) is

$$\frac{\partial}{\partial \eta} \; \dot{\underline{x}} \; = \; \frac{\partial f}{\partial x} \underline{s} + \frac{\partial f}{\partial u} \; \frac{\partial u}{\partial x} \underline{s} + \frac{\partial f}{\partial \eta} \tag{2.36}$$

where all partials exist by definition of $\underline{f}(\cdot)$ and $\underline{u}(\cdot)$ and are defined relative to the nominal $\underline{x}_n(t)$, $\underline{u}_n(t)$ and η_n. Also since the solution of (2.35), $\underline{x}(t, \eta)$, is continuous in t and η, $\frac{\partial \dot{x}}{\partial \eta} = \dot{s}$. From s_3), (2.36) thus becomes

$$\dot{\underline{s}} = \frac{\partial f}{\partial x} \; \underline{s} + \frac{\partial f}{\partial u} \; K(t) \; \underline{s} \; + \; \frac{\partial f}{\partial \eta} \quad ; \quad \underline{s}(t_1) = 0 \tag{2.37}$$

where $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial \eta}$ are integrable in t and $\frac{\partial f}{\partial u}$ is essentially bounded (in L^∞). Note that for the product $\frac{\partial f}{\partial u} \cdot K$ to be integrable, Hölders inequality ([33], [34]) implies that at least one of the functions must be in L^∞. Since (2.34) is quadratic, it was necessary to assume L^∞ for both.

Equations (2.34) and (2.37) define a subsidiary minimization problem which determins the optimal gain function $\overline{K}(t)$ relative to assumptions s_1) through s_4). The advantages of this formulation are that the problem is linear and quadratic and that it is independent of the actual parameter values. From (2.29) the desired control input to (2.18) is

$$\underline{u}(t, \underline{x}) = \underline{u}_1(t) \; + \; \overline{K}(t) \; \underline{x}(t) \tag{2.38}$$

or, by application of the nominal equivalence condition on $\underline{u}_1(t)$,

$$\underline{u}(t, \underline{x}) = \underline{u}_n(t) \; + \; \overline{K}(t) \; [\; \underline{x}(t) - \underline{x}_n(t) \;] \; . \tag{2.39}$$

It is shown in Appendix A that compensators of this type are related to classical input compensators for linear systems.

In summary, the problem has been reduced to that of minimizing the cost functional (2.34) subject to the system equation (2.37) over all (nxr) matrices $K(t)$ of essentially bounded functions. If, in section 2.3, the elements of $K(t)$ are corresponded to $\underline{u}(t)$ and similarly $\underline{s}(t)$ to $\underline{x}(t)$, then it is easily seen that conditions m) through p) are satisfied. Theorem 2.2 thus gives necessary conditions for the minimizing gain $\overline{K}(t)$. It should be noted that a solution to this problem cannot be obtained by application of the maximum principle [32] or standard variational techniques [30]. For these methods to be valid, the system and cost matrices must be continuous instead of being essentially bounded or integrable as assumed above.

Chapter 3
Reduced Sensitivity Solution

3.1 Introduction

The problem of minimizing (2.34) subject to (2.37) will be deferred until Chapter 4. In this chapter the main consideration is the sensitivity equation

$$\dot{\underline{s}} = A\,\underline{s} + BK\,\underline{s} + \underline{g} \quad ; \; s(t_1) = 0 \tag{3.1}$$

where the A, B and g coefficients respectively correspond to those of (2.37) and are assumed continuous in time. Two gain functions K(t) will be examined which in some sense reduce the closed loop sensitivity $\underline{s}_c(t)$ given by (3.1) relative to the open loop sensitivity $\underline{s}_o(t)$ given also by (3.1) but with $K(t) \equiv 0$ on $[t_1, t_2]$. One function is a decoupling control and the other is the regulator gain.

3.2 Model Following Control

<u>Sensitivity Bounds</u>

Rewrite equation (3.1) in the following form

$$\dot{\underline{s}} = H\,\underline{s} + \underline{g} \quad ; \; s(t_1) = 0 \tag{3.2}$$

where the (nxn) system matrix H is given by

$$H = A + BK \quad . \tag{3.3}$$

It is desirable to choose K such that

$$H = \begin{bmatrix} -\alpha_1 & & 0 \\ & \ddots & \\ 0 & & -\alpha_n \end{bmatrix} \quad ; \; \alpha_i > 0 \tag{3.4}$$

for then (3.2) can be analyzed as n uncoupled scalar equations. If the control dimension r is greater than n and B is at least of rank n, then K can be realized by

$$K = B^T \, [BB^T]^{-1} \; [H - A] \quad .$$

But if $r < n$, as is usually the case, techniques such as decoupling [35] or model following [36] must be employed. A simple model following technique is described in the next section.

In what follows, it is assumed that H has been decoupled as in (3.4) with the positive scalars α_i being constant. Also, to obtain realistic bounds on the sensitivities, a disturbance term will be added to (3.1), i.e.

$$\dot{\underline{s}} = A\,\underline{s} + BK\,\underline{s} + \underline{g} + BK\,\underline{d} \quad ; \quad \underline{s}(t_1) = 0 \tag{3.5}$$

where $\underline{d}$ represents measurement errors. State of system disturbances are included in $\underline{g}$. Thus (3.5) becomes

$$\dot{\underline{s}} = H\,\underline{s} + \underline{g} + [H - A]\,\underline{d} \quad ; \quad \underline{s}(t_1) = 0$$

or

$$\dot{s}_i = \alpha_i s_i - \alpha_i d_i + g_i - (A\underline{d})_i \quad . \tag{3.6}$$

Then for $i = 1,\ldots,n$ and $t \in [t_1, t_2]$

$$|s_i(t)| \leq M_i\,(t, \alpha_i) \tag{3.7}$$

where

$$M_i(t, \alpha_i) = \frac{1}{\alpha_i}\left[\,1 - e^{-\alpha_i(t-t_1)}\,\right][b_i + \alpha_i\,\bar{d}_i]$$

$$b_i = \sup_t\,[g_i - (A\,\underline{d})_i] \quad \text{and} \quad \bar{d}_i = \sup_t\,[d_i]\ .$$

For a fixed $t' \in [t_1, t_2]$ the bound behaves as follows relative to variations in α_i

$$\lim_{\alpha_i \to 0} M(t', \alpha_i) = b_i(t' - t_1) \quad \text{and} \quad \lim_{\alpha_i \to \infty} M(t', \alpha_i) = \bar{d}_i \quad .$$

It is thus seen that the sensitivity bound approaches that of the measurement noise for large system (feedback) gains. For a fixed α'_i and t_2 unbounded

$$\lim_{t \to t_1} M(t, \alpha'_i) = 0 \qquad \text{and} \qquad \lim_{t \to \infty} M(t, \alpha'_i) = {}^{b_i}/_{\alpha'_i} + \bar{d}_i \quad .$$

Thus, depending on the length of the interval $[t_1, t_2]$, α_i can be determined from a tradeoff between the relative magnitudes of the state, b_i, and measurement, $\bar{d}_i$, disturbances.

Optimal Model Following

The problem considered in this section is that of determining the feedback gain K such that the system given by (3.1) performs as (3.2) with a fixed system matrix H. Treating the parameter variation term $\underline{g}(t)$ as a disturbance, it is desired to choose a control function $\underline{u}(t) = K(t)\,\underline{s}(t)$ such that

$$\dot{\underline{s}} = A\,\underline{s} + B\,\underline{u} \quad ; \quad \underline{s}(t_1) = \underline{s}_1 \tag{3.8}$$

performs as

$$\dot{\underline{s}} = H\,\underline{s} \quad ; \quad \underline{s}(t_1) = \underline{s}_1 \tag{3.9}$$

To this end the following cost functional is employed

$$J(u) = \frac{1}{2} \int_{t_1}^{t_2} [(\dot{\underline{s}} - H\underline{s})^T Q(\dot{\underline{s}} - H\underline{s}) + \underline{u}^T R \underline{u}]\, dt \tag{3.10}$$

where Q and R are positive definite weighting matrices and $\underline{s}(t)$ is given by (3.8). Letting E = A-H, (3.10) becomes

$$J(u) = \frac{1}{2} \int_{t_1}^{t_2} [(E\underline{s} + B\underline{u})^T Q(E\underline{s} + B\underline{u}) + \underline{u}^T R \underline{u}]\, dt. \tag{3.11}$$

The minimization of (3.11) subject to (3.8) is similar to the regulator problem discussed in [10] and [11]. Using standard variational techniques, the optimal control is

$$\underline{u}(t) = \bar{R}^{-1} B^T [\underline{p} - Q\,E\,\underline{s}] \tag{3.12}$$

where $\bar{R} = [B^T Q B + R]$ with $\underline{p}(t)$ and $\underline{s}(t)$ satisfying

$$\underline{\dot{p}} = [\ -A^T + E^T Q B \tilde{R}^{-1} B^T] \underline{p} + E^T[I + Q B \tilde{R}^{-1} B^T] Q E \underline{s}$$
$$\underline{\dot{s}} = [\ A - B\tilde{R}^{-1} B^T Q E] \underline{s} + B\tilde{R}^{-1} B^T \underline{p} \tag{3.13}$$

$$\underline{s}(t_1) = 0 \quad ; \quad \underline{p}(t_2) = 0 \quad .$$

Setting $\underline{p}(t) = \underline{\mathcal{R}}(t)\ \underline{s}(t)$ in (3.13) a Riccati equation can be obtained for $\mathcal{R}$ as described in [11]. Thus the desired model following gain is

$$K(t) = [B^T Q B + R]^{-1} B^T [\mathcal{R}(t) + Q(H - A)] \quad . \tag{3.14}$$

3.3 Regulator and Stabilizing Controls

In this section the sensitivity problem discussed in section 2.4 will be slightly restated such that a standard solution technique can be employed to obtain the feedback gain. The cost functional will remain unchanged, i.e.

$$J = \frac{1}{2} \underline{s}^T(t_2) D\underline{s}(t_2) + \frac{1}{2} \int_{t_1}^{t_2} [\underline{s}^T Q \underline{s} + \underline{s}^T K^T R K \underline{s}] dt \tag{3.15}$$

where all matrices are assumed continuous. The first two terms represent final value and mean square sensitivity measures whereas the last term limits the amount of feedback. In the sensitivity equation (3.1) the parameter error forcing term, $\underline{g}(t)$, will be treated as a disturbance which effectively produces initial condition errors along the trajectory. Thus (3.1) can be replaced by

$$\underline{\dot{s}} = A \underline{s} + B K \underline{s} \quad ; \quad \underline{s}(t_1) = \underline{s}_1 \tag{3.16}$$

where $\underline{s}_1$ is arbitrary. This method of treating disturbances has been used in [37] to generate linear feedback controls for optimal nonlinear systems. The minimization of (3.15) subject to (3.16) was discussed in [11]. The solution is

$$K(t) = -R^{-1} B^T P(t) \tag{3.17}$$

where the (nxn) matrix P satisfies

$$-\dot{P} = PA + A^T P - PBR^{-1} B^T P + Q \quad ; \quad P(t_2) = 0 \,. \tag{3.18}$$

The remaining question is what effect does the approximation of the parameter error forcing term by the arbitrary initial condition vector have on the first order sensitivity given by (3.1)? Let the open loop sensitivity, $\underline{s}_o$, be given by

$$\dot{\underline{s}}_o = A\,\underline{s}_o + \underline{q} \quad ; \quad \underline{s}_o(t_1) = 0 \tag{3.19}$$

and similarly the closed loop sensitivity by

$$\dot{\underline{s}}_c = A\,\underline{s}_c + B\,K\,\underline{s}_c + \underline{q} \,; \; \underline{s}_c(t_1) = 0 \tag{3.20}$$

where K is computed from (3.17) and (3.18). The following theorem, which is similar in structure to that of [14] for linear optimal systems, states that a sensitivity reduction does take place with the use of the regulator gain.

<u>Theorem 3.3</u>: The regulator gain defined by (3.17) and (3.18) when employed as a feedback control (2.29) causes a sensitivity reduction as follows:

$$\int_{t_1}^{t'} \underline{s}_c^{\,T} K^T RK\,\underline{s}_c\,dt \leq \int_{t_1}^{t'} \underline{s}_o^{\,T} K^T R\,K\,\underline{s}_o\,dt \tag{3.21}$$

where the open loop, $\underline{s}_o$, and closed loop $\underline{s}_c$, sensitivities are given by (3.19) and (3.20) and $t' \in (t_1, t_2)$.

Proof: Integrating (3.19) and (3.20), it is seen that

$$\underline{s}_o(t) = \underline{s}_c(t) + \underline{m}(t) \tag{3.22}$$

where

$$\underline{m}(t) = -\int_{t_1}^{t} \phi(t,\tau)\,B\,K\,\underline{s}_c\,d\tau \tag{3.23}$$

and $\phi(t, \tau)$ is the transition matrix corresponding to A, i.e.

$$\dot{\phi} = A\,\phi \quad ; \quad \phi\,(t, t) = I \quad . \tag{3.24}$$

Let $N = K^T R K$, an (nxn) positive semi-definite matrix, and form using (3.22)

$$\underline{s}_o^T N \underline{s}_o - \underline{s}_c^T N \underline{s}_c = 2\underline{m}^T N \underline{s}_c + \underline{m}^T N \underline{m} \quad .$$

Thus a necessary and sufficient condition for (3.21) is that

$$\int_{t_1}^{t'} [\, 2\, \underline{m}^T N \underline{s}_c + \underline{m}^T N \underline{m} \,]\, dt > 0 \quad . \tag{3.25}$$

Substituting (3.17) into (3.23) gives

$$\underline{m}(t') = \int_{t_1}^{t'} \phi(t, \tau)\, C(\tau)\, P(\tau)\, \underline{s}_c(\tau)\, d\tau \tag{3.26}$$

where $C = B R^{-1} B^T$. A differential equation for $\underline{m}(t)$ can easily be obtained from (3.26) as

$$\dot{\underline{m}} = A\,\underline{m} + C(t)\, P(t)\, \underline{s}_c(t) \quad ; \quad \underline{m}(t_1) = 0 \quad . \tag{3.27}$$

Also multiplying (3.18) by $\underline{m}^T$ and then $\underline{m}$ yields

$$-\frac{d}{dt}\,[\underline{m}^T P\, \underline{m}] = -2\, \underline{m}^T P\, C\, P\, \underline{s}_c - \underline{m}^T P\, C\, P\, \underline{m} + \underline{m}^T Q\, \underline{m}$$

where (3.27) was substituted for A $\underline{m}$. Now the above can be integrated between t_1 and t' to give

$$-\underline{m}^T(t')\, P(t')\, \underline{m}(t') = -\int_{t_1}^{t'} [2\, \underline{m}^T PCP\, \underline{s}_c + \underline{m}^T PCP\, \underline{m}]\, dt + \int_{t_1}^{t'} \underline{m}^T Q\, \underline{m}\, dt \ .$$

But since $P(t')$ is positive semi-definite $\forall\ t' \in (t_1, t_2)$ and Q is positive definite, the above implies that

$$\int_{t_1}^{t'} [\, 2\, \underline{m}^T\, PCP\, \underline{s}_c + \underline{m}^T\, PCP\, \underline{m}]\, dt > 0 \quad . \tag{3.28}$$

Also by (3.17) and (3.26)

$$K^T\, R\, K = P\, B\, R^{-1}\, R\, R^{-1}\, B^T P = P\, C\, P$$

and thus (3.28) implies (3.21) which completes the proof. Note that when $\underline{m}(t)$ is nonzero on (t_1, t'), (3.28) and consequently (3.21) are strict inequalities. This will usually occur for nonzero $\underline{q}(t)$ if (3.20) is controllable.

An interesting corollary can be obtained from Theorem 3.3 which explicitly relates stability and sensitivity reduction. To this end assume that the system matrix A in (3.1) is constant and stable (negative eigenvalues). Then it was shown in [38] that the equation

$$-\, I = A^T\, Y + Y\, A \tag{3.29}$$

has a unique solution Y which is symmetric and positive semi-definite. Define the feedback gain by

$$K = B^T\, Y \quad . \tag{3.30}$$

Corollary 3.3: The use of the gain defined by (3.30) and (3.29) as a feedback control causes a sensitivity reduction as follows:

$$\int_{t_1}^{t'} \underline{s}_c^{\,T}\, K^T\, K\, \underline{s}_c\, dt < \int_{t_1}^{t'} \underline{s}_o^{\,T}\, K^T\, K\, \underline{s}_o\, dt \tag{3.31}$$

where the open loop, $\underline{s}_o$, and closed loop, $\underline{s}_c$, sensitivites are given by (3.19) and (3.20) with A a constant, stable matrix and $t' \in (t_1, t_2)$.

Proof: The method of proof is similar to that of Theorem 3.3 and will thus not be repeated.

It should be noted that Porter obtained similar results in [8] using frequency domain techniques; however, it was necessary to assume that the time interval (t_1, t_2) of control was unbounded. Thus Corollary 3.3 effectively extends those results.

Chapter 4

Minimum Sensitive Gain Feedback Control

4.1 Introduction

In this chapter the explicit problem of determining a feedback gain matrix which minimizes (2.34) subject to (2.37) will be considered. An additional assumption of continuity in time will be placed on the original system functions so that standard second order conditions can be applied. The minimization problem examined in the remainder of this chapter is for completeness restated below.

The closed loop system dynamics are described by

$$\dot{\underline{x}} = \underline{f}(t, \underline{x}, \underline{u}(t,\underline{x}), \underline{\eta}) \; ; \quad \underline{x}(t_1) = \underline{x}_{n1} \qquad (4.1)$$

where $\underline{x}$, $\underline{u}(\cdot)$ and $\underline{\eta}$ have dimensions n, r and m respectively and $\underline{f}(\cdot)$ is continuous in t and C^1 WRT $\underline{x}$, $\underline{u}(\cdot)$ and $\underline{\eta}$. The control function is given by

$$\underline{u}(t,\underline{x}) = \underline{u}_1(t) + K(t)\,\underline{x}(t) \qquad (4.2)$$

where $\underline{u}_1(t)$ is determined such that

$$\underline{u}(t, \underline{x}_n) = \underline{u}_n(t) \qquad (4.3)$$

with $\underline{x}_n(t)$ and $\underline{u}_n(t)$ being the nominal solution of (4.1) for the parameter $\underline{\eta} = \underline{\eta}_n$. The problem is to determine the (rxn) gain matrix K(t) such that, for small variations $\underline{\Delta\eta}$ from the nominal parameter $\underline{\eta}_n$, the actual trajectory $\underline{x}(t)$ remains close to $\underline{x}_n(t)$ over the original optimization interval. It will initially be assumed that $\underline{\eta}$ is known to within a scalar constant, i.e. $\underline{\eta} = \eta_a \hat{\underline{\eta}}$ where η_a is an unknown magnitude operating through a known direction $\hat{\underline{\eta}}$. The results will later be extended to multiple parameter variations. From section 2.4, the first order sensitivity vector $\underline{s}(t)$ relative to η_a is described by

$$\dot{\underline{s}} = A(t)\,\underline{s} + B(t)\,K(t)\,\underline{s} + \underline{g}(t)\ ; \quad \underline{s}(t_1) = \underline{s}_0 \tag{4.4}$$

where A, B and $\underline{g}$ represent the partial derivatives of (4.1) WRT $\underline{x}$, $\underline{u}(\cdot)$ and $\underline{\eta}$ respectively evaluated along the nominal. The initial value of the sensitivity vector, $\underline{s}_0$, will normally be zero since the parameter will usually not affect the initial state $\underline{x}_{n1}$. In what follows it is assumed that the original problem (4.1) is defined over a fixed time interval [0,T].

The sensitivity cost function employed in this chapter is similar to that given by (2.34). Two measures of trajectory sensitivity are

$$\text{mean square} = \frac{1}{2}\int_0^T \underline{s}^T Q\,\underline{s}\,dt$$

$$\text{final value} = \frac{1}{2}\,\underline{s}^T(T)\,D\,\underline{s}(T)$$

where Q and D are positive semi-definite matrices which are continuous in time. The system error is limited by restricting the amount of feedback $K(t)\,\underline{x}(t)$ or equivalently $K(t)\,\underline{s}(t)$. This restriction can be included in the cost by the addition of one of the following functions

$$F_1 = \frac{1}{2}\int_0^T \left[\sum_{i=1}^{r}\sum_{j=1}^{n} R_{ij}\,K_{ij}^2\,s_j^2\right] dt$$

$$F_2 = \frac{1}{2}\int_0^T \left[\underline{s}^T K^T R\,K\,\underline{s}\right] dt$$

where $R_{ij} > 0\ \ \forall i,j$ and R is an (rxr) positive definite symmetric matrix, both of which are continuous in time. The term F_1 restricts each state feedback component whereas F_2 restricts each control component. The use of either F_1 or F_2 depends on the number of unknown parameters and will be discussed further in the sequel. For scalar parameter variations, F_1 can be combined with the trajectory sensitivity measures to yield the following cost functional

$$J(K) = \frac{1}{2}\,\underline{s}^T(T)\,D\,\underline{s}(T) + \frac{1}{2}\int_0^T [\,\underline{s}^T\,Q\,\underline{s} + \sum_{i=1}^{r}\sum_{j=1}^{n} R_{ij}\,K_{ij}^2\,s_j^{\,2}]\,dt \quad (4.5)$$

which effectively trades off the cost of feedback for reductions in trajectory sensitivity. The problem is thus to determine K(t) such that (4.5) is minimized subject to (4.4).

The following sections contain necessary and sufficient conditions for the existence of the minimum sensitive feedback gain. In addition, the relationships between minimum sensitive control and least square parameter estimation are discussed in Appendix C.

4.2 Necessary Conditions for Minimum Sensitivity with State Feedback Cost

Necessary conditions for the problem posed in section 4.1 can be obtained by straightforward application of variational methods given in [30] and [40]. The Hamiltonian is defined as follows:

$$H_1(t, \underline{s}, K, \underline{p}) = -\frac{1}{2}\,\underline{s}^T Q\,\underline{s} - \frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{n} R_{ij}\,K_{ij}^2\,s_j^{\,2} + \underline{p}^T[\,A\,\underline{s} + B\,K\,\underline{s} + \underline{g}\,] \quad . \qquad (4.6)$$

The adjoint equation is given by

$$\dot{\underline{p}} = -\nabla_s H_1 = Q\,\underline{s} + \frac{1}{2}\nabla_s [\sum_{i=1}^{r}\sum_{j=1}^{n} R_{ij}\,K_{ij}^2\,s_j^{\,2}] - \underline{p}^T A - \underline{p}^T BK \quad (4.7)$$

$$\text{with } \underline{p}(T) = -D\,\underline{s}(T) \quad .$$

The optimal gain K(t) is found by

$$\frac{\partial H_1}{\partial K_{ij}} = -R_{ij}\,K_{ij}\,s_j^{\,2} + \sum_{\ell=1}^{n} p_\ell\,B_{\ell i}\,s_j = 0$$

which yields for $s_j \neq 0$

$$K_{ij} = \frac{1}{R_{ij}\, s_j} \sum_{\ell=1}^{n} p_\ell\, B_{\ell i} \quad . \tag{4.8}$$

The general form of the canonical equations can be found by substituting (4.8) into (4.7) and (4.4). The $j\underline{th}$ component of (4.7) is as follows

$$\dot{p}_j = \sum_{\ell=1}^{n} Q_{j\ell}\, s_\ell + \sum_{\ell=1}^{r} R_{\ell j}\, K_{\ell j}^2\, s_j - \sum_{i=1}^{n} p_i\, A_{ij}$$

$$- \sum_{i=1}^{n} \sum_{\ell=1}^{r} p_i\, B_{i\ell} K_{\ell j}$$

$$= \sum_{\ell=1}^{n} Q_{j\ell}\, s_\ell - \sum_{i=1}^{n} p_i\, A_{ij}$$

$$+ \sum_{\ell=1}^{r} \frac{K_{\ell j}}{s_j} \left[R_{\ell j}\, K_{\ell j}\, s_j^{\,2} - \sum_{i=1}^{n} p_i\, B_{i\ell}\, s_j \right]$$

where the bracket term is zero from (4.8). Thus (4.7) becomes

$$\underline{\dot{p}} = - A^T\, \underline{p} + Q\, \underline{s} \quad ; \quad \underline{p}(T) = - D\, \underline{s}(T) \quad . \tag{4.9}$$

Similarly the $j\underline{th}$ component of (4.4) is

$$\dot{s}_j = \sum_{i=1}^{n} A_{ij}\, s_i + \sum_{\ell=1}^{r} \sum_{i=1}^{n} B_{j\ell}\, K_{\ell i}\, s_i + g_j$$

$$= \sum_{i=1}^{n} A_{ji}\, s_i + \sum_{\ell=1}^{r} B_{j\ell} \sum_{i=1}^{n} \frac{1}{R_{\ell i}} \sum_{m=1}^{n} p_m\, B_{m\ell} + g_j \quad .$$

Defining
$$V_{\ell y} = \begin{cases} \sum_{i=1}^{n} \frac{1}{R_{\ell i}} & \ell = y \\ 0 & \ell \neq y \end{cases} \tag{4.10}$$

The above becomes

$$\dot{s}_j = \sum_{i=1}^{n} A_{ji}\, s_i + \sum_{\ell=1}^{r} \sum_{y=1}^{r} \sum_{m=1}^{n} B_{j\ell}\, V_{\ell y}\, B_{my}\, p_m + g_j$$

or in vector notation

$$\dot{\underline{s}} = A\,\underline{s} + B\,V\,B^T\,\underline{p} + \underline{g}\ ; \quad \underline{s}(0) = 0 \quad . \tag{4.11}$$

Thus the canonical equations (4.9) and (4.11) are time invariant whenever the sensitivity equations and cost matrices are independent of time. The linearity therefore allows a closed form solution for the gain terms given by (4.8). Note that since $\underline{s}_0 = 0$ any value of K(0) will satisfy the optimality conditions. In practice, however, an initial bound must be determined for K(t).

The Legendre condition is obtained from (4.6) as

$$\frac{\partial^2 H_1}{\partial K_{ij}^{\,2}} = -R_{ij}\, s_j^{\,2} < 0 \tag{4.12}$$

$$\frac{\partial^2 H_1}{\partial K_{ij}\, \partial K_{\ell m}} = 0 \qquad i, j \neq \ell, m \quad .$$

The Weierstrass necessary condition is implied by (4.12) when the extremal is nonsingular (reference [30]).

It should be noted that even though a solution to the canonical equations (4.9) and (4.11) may exist $\forall\ t \in [0, T]$, the gain given by (4.8) may not exist as a solution to the optimization problem. This is because the sensitivity vector (4.11) may be such that $s_j(t') = 0$ for some j and $t' \in (0, T]$. When this occurs the extremal becomes singular,

$$\det \left| \frac{\partial^2 H_1}{\partial K_{ij}\, \partial K_{\ell m}} \right| = 0$$

and sufficiency conditions are not satisfied. This will be discussed in greater detail in section 4.5.

4.3 Necessary Conditions for Minimum Sensitivity with Control Feedback Cost

In a manner similar to that of the previous section, necessary conditions can be obtained for the problem of minimizing (4.5) with F_1 replaced by F_2. The Hamiltonian is

$$H_2(t, \underline{s}, K, \underline{p}) = -\frac{1}{2}\underline{s}^T Q\underline{s} - \frac{1}{2}\underline{s}^T K^T RK\underline{s} + [A\underline{s} + BK\underline{s} + \underline{g}]^T \underline{p} \quad . \tag{4.13}$$

The adjoint equation is

$$\dot{\underline{p}} = -\nabla_s H_2 = Q\underline{s} + K^T RK\underline{s} - A^T \underline{p} - K^T B^T \underline{p} \tag{4.14}$$

$$\text{with } \underline{p}(T) = -D\,\underline{s}(T) \quad .$$

The gain $K(t)$ is determined by

$$\frac{\partial H_2}{\partial K_{ij}} = [\sum_{\ell=1}^{r} \sum_{m=1}^{n} R_{i\ell} K_{\ell m} s_m - \sum_{m=1}^{n} p_m B_{mi}] s_j = 0$$

or in vector form when $s_j \neq 0$ for some j

$$-RK\underline{s} + B^T\underline{p} = 0 \quad . \tag{4.15}$$

Substituting the above in (4.14) and (4.4) results in the following canonical equations,

$$\dot{\underline{p}} = -A^T\underline{p} + Q\underline{s} \quad ; \quad \underline{p}(T) = -D\underline{s}(T) \tag{4.16}$$

$$\dot{\underline{s}} = A\underline{s} + BR^{-1}B^T\underline{p} + \underline{g} \; ; \quad \underline{s}(0) = 0 \tag{4.17}$$

which are easily solved when the system and cost matrices are time invariant. Equation (4.15) however does not explicitly determine the optimal gain K(t) since only r conditions are given for the (rxn) matrix. This occurs because the cost function weights only the r dimensional control elements. Thus additional conditions are required to completely determine the gain matrix. It should be noted that equations (4.15) through (4.17) with $\underline{g}(t) \equiv 0$ and $\underline{s}(0) \neq 0$ are equivalent to the solution of the regulator problem described in [11]. The additional condition imposed on that problem to uniquely determine the gain function is that the differential (Riccati) equation satisfied by K(t) must hold for all $\underline{s}_0$. This cannot be done when $\underline{g}(t) \neq 0$ since (4.16) and (4.17) then generate a particular sensitivity vector $\underline{s}(t)$ from $\underline{s}(0) = 0$.

4.4 Extension to Multiparameter Variations

It will initially be assumed that the original system equation (4.1) is linear with a single input and in phase variable canonical form (reference [39]). Thus (4.1) and (4.2) become

$$\dot{\underline{x}} = A\,\underline{x} + BK\,\underline{x} + B\,u_1(t) \tag{4.18}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & 0 \\ \vdots & & & \ddots & & 0 \\ 0 & \cdot & \cdot & \cdot & 0 & 1 \\ \eta_1 & \cdot & \cdot & \cdot & \cdot & \eta_n \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

The dynamics are therefore a function of n parameters $\eta_1,\ldots,\eta_n$. Corresponding to each parameter η_i, a sensitivity equation can be determined as follows:

$$\dot{\underline{s}}_i = A\underline{s}_i + BK\,\underline{s}_i + \underline{g}_i \tag{4.19}$$

where $\underline{g}_i^T = [\,0,\ldots,0,\,x_i]$ with x_i the ith component of the nominal solution of (4.18). Also, a control feedback cost function (section 4.3) can be formulated to minimize the effects of the parameter η_i.

The optimal gain must then satisfy

$$K \underline{s}_i = R_i^{-1} B^T \underline{p}_i \tag{4.20}$$

where $\underline{s}_i$ and $\underline{p}_i$ are determined from equations similar to (4.16) and (4.17). Define the sensitivity and adjoint matrices S and P as follows:

$$S = [\ \underline{s}_1 \ \cdots\cdots \ \underline{s}_n\ ]$$

$$P = [\ R_1^{-1} B^T \underline{p}_1 \ \cdots\cdots \ R_n^{-1} B^T \underline{p}_n\ ] \quad .$$

Then equation (4.20) when combined $\forall_i$ becomes

$$K S = P \quad . \tag{4.21}$$

If the sensitivity vectors $\underline{s}_i(t)$ are linearly independent $\forall\ t \in (0,T]$, the multiparameter minimum sensitivity feedback gain is given by

$$K = P S^{-1} \quad . \tag{4.22}$$

Since $\underline{s}_i(0) = 0$, K must be bounded at t = 0.

The above analysis need not be restricted to single input linear systems in phase variable canonical form, although at least n parameters must be involved and the sensitivity vectors must be linearly independent. If the number of parameters is less than n, a combination of the feedback costs F_1 and F_2 can be employed to determine the optimal gain.

The existence of the optimal gain (4.22) is determined by the linear independence of the sensitivity vectors $\underline{s}_i(t)$. This is because, from section 4.3, the gain equation (4.20) holds if only one component of $\underline{s}_i(t)$ is nonzero. Thus if the matrix S is nonsingular $\forall\ t \in (0,T]$, (4.20) holds $\forall i$.

4.5 Sufficient Conditions for Minimum Sensitivity with State Feedback Cost

The possibility of nonexistent solutions to the minimum sensitivity problem with state feedback cost was briefly discussed in section 4.2. In this section, the material in Appendix B is used to strengthen the necessary conditions to obtain the following existence theorem.

Theorem 4.5: The gain matrix K(t) given by (4.8), (4.9) and (4.11) exists on (0,T] as a minimum of (4.5) subject to (4.4) if

$$s_j^2(t) > 0 \quad \forall\, t \in (0,\, t] \tag{4.23}$$

where $s_j(t)$ is the jth component of the solution to (4.11).

Proof: From Appendix B the existence of a weak minimum requires satisfaction of conditions 1), 2), 3) and 5). Since (4.8), (4.9) and (4.11) are the extremal equations, condition 1) is automatically satisfied. In addition, (4.12) and (4.23) imply conditions 2) and 3) and thus it remains to check for conjugate points. The (nxm) matrix elements of (B.5) are determined from (4.6) through (4.11) to be

$$\begin{bmatrix} \bar{H}_{ps} & \bar{H}_{pp} \\ -\bar{H}_{ss} & -\bar{H}_{sp} \end{bmatrix} = \begin{bmatrix} A & BVB^T \\ Q & -A^T \end{bmatrix} . \tag{4.24}$$

From the definitions of the cost function (4.5) and of the (rxr) matrix V (4.10) the matrices $\bar{H}_{pp}$, $-\bar{H}_{ss}$ and D are positive semi-definite on (0,T]. Thus by Theorems B.1 and B.2, no conjugate points exist and the proof is complete.

From the above theorem, the existence of the minimum sensitive gain is determined mainly by (4.23) which is a somewhat strong condition and definitely not satisfied for arbitrary cost parameters D, Q and R in (4.5) and arbitrary functions $\underline{g}(t)$ in (4.11). This is, however, the price of achieving linearity of the canonical equations (4.9) and (4.11). For a given system, cost function and nominal trajectory, these equations can easily be solved to determine if (4.23) is satisfied. If not, the nonsingular approximate problem formulated in the next section can be employed to obtain the optimal gain.

In some cases when (4.23) does not hold, the canonical equations (4.9) and (4.11) can be adjusted such that (4.23) is satisfied. The form of these equations closely resembles that of the canonical equations for the regulator with $\underline{g} = 0$. Since the regulator gain holds for arbitrary initial conditions, it is plausible that the magnitude of the initial sensitivity vector in (4.11), $\underline{s}(0)$, could be increased from zero such that the effect of $\underline{g}(t)$ becomes increasingly less important in the solution for K(t). In order to do this the sign of $\underline{s}(0)$ must be consistent with that of the trajectory generated by $\underline{g}(t)$. The initial sensitivity vector $\underline{s}(0)$ is thus treated as a parameter in the optimization problem which may be adjusted such that sufficiency conditions are satisfied. This method was successfully employed in the example discussed in Chapter 5.

4.6 A Nonsingular Sensitivity Problem

Necessary Conditions

The results of the previous section indicate that singular solutions of the minimum sensitivity problem with state feedback cost are the major cause for failure of the existence conditions. The problem will be reformulated in this section such that all extremals are nonsingular. As a consequence of this, the canonical equations become nonlinear and must be solved either by approximation or iterative techniques.

Examination of (4.23) and (4.8) reveals that singularities in the optimal gain are synonymous with singular extremals. The cost function (4.5) will therefore be modified to include a penality term for large feedback gains as follows:

$$J(K) = \frac{1}{2}\,\underline{s}^T(T)\,D\,\underline{s}(T) + \frac{1}{2}\int_0^T [\,\underline{s}^T Q \underline{s} + G]\,dt$$

with (4.25)

$$G = \sum_{i=1}^{r}\sum_{j=1}^{n} (R_{ij}\,K_{ij}^2\,s_j^2 + E_{ij}\,K_{ij}^2)$$

and $E_{ij} > 0 \; \forall \; i,j$. The Hamiltonian for the problem of minimizing (4.25) subject to (4.4) is

$$H_3 \, (t, \underline{s}, K, p) = -\frac{1}{2}\underline{s}^T Q\underline{s} - G + \underline{p}^T [A\underline{s} + BK\underline{s} + \underline{g}]. \quad (4.26)$$

The adjoint equation is

$$\dot{\underline{p}} = -\nabla_s H_3 = Q\underline{s} + \frac{1}{2}\nabla_s G - \underline{p}^T A - \underline{p}^T BK \quad (4.27)$$

with

$$\underline{p}(T) = -D\,\underline{s}(T) \quad .$$

The ℓth component of the feedback term in (4.27) is evaluated as

$$\frac{1}{2}\frac{\partial G}{\partial s_\ell} = \sum_{i=1}^{r} R_{i\ell} K_{i\ell}^2 s_\ell \quad .$$

Equation (4.27) thus becomes

$$\dot{\underline{p}} = -[A + BK]^T \underline{p} + [Q + X(K)] \underline{s} \quad (4.28)$$

with the (nxn) matrix X defined as

$$X_{\ell m} = \begin{cases} \sum_{i=1}^{r} R_{i\ell} K_{i\ell}^2 & \ell = m \\ 0 & \ell \neq m \end{cases} \quad .$$

The optimality condition is obtained from equation (4.26)

$$\frac{\partial H_3}{\partial K_{ij}} = -R_{ij} K_{ij} s_j^2 - E_{ij} K_{ij} + \sum_{\ell=1}^{n} p_\ell B_{\ell i} s_j \quad . \quad (4.29)$$

Setting the above to zero and solving for K_{ij} yields

$$K_{ij} = \frac{s_j}{R_{ij} s_j^2 + E_{ij}} \cdot \sum_{\ell=1}^{n} p_\ell B_{\ell i} \quad . \quad (4.30)$$

The above equation can be used to eliminate the gain variables in the canonical equations as follows. From (4.28), define the n dimensional vectors

$$\underline{M} = - K^T B^T \underline{p} + X(K)\, \underline{s}$$

or componentwise

$$M_j = - \sum_{\ell=1}^{n} \sum_{i=1}^{r} p_\ell\, B_{\ell i}\, K_{ij} + \sum_{i=1}^{r} R_{ij}\, K_{ij}^2\, s_j \quad .$$

Substituting (4.30) into the above yields

$$M_j = - \frac{E_{ij}\, s_j}{[R_{ij} s_j^2 + E_{ij}]^2} \cdot \left[\sum_{\ell=1}^{n} p_\ell\, B_{\ell i} \right]^2 \quad . \tag{4.31}$$

Thus equation (4.28) is

$$\dot{\underline{p}} = - A^T \underline{p} + Q\, \underline{s} + \underline{M}(\underline{s}, \underline{p}) \tag{4.32}$$

with $\underline{p}(T) = - D\, \underline{s}(T)$ and $\underline{M}$ given by (4.31). Using (4.30) and (4.4), the $j\underline{th}$ component of $(BK\underline{s})$ is

$$(BK\underline{s})_j = \sum_{\ell=1}^{r} \sum_{m=1}^{n} B_{j\ell}\, K_{\ell m}\, s_m$$

$$= \sum_{\ell=1}^{r} B_{j\ell} \sum_{m=1}^{n} \frac{s_m^2}{R_{\ell m}\, s_m^2 + E_{\ell m}} \cdot \sum_{i=1}^{n} p_i\, B_{i\ell} \quad .$$

Define the (rxr) matrix Z as

$$Z_{\ell y} = \begin{cases} \sum_{m=1}^{n} \dfrac{s_m^{\,2}}{R_{\ell m}\, s_m^{\,2} + E_{\ell m}} & \ell = y \\ 0 & \ell \neq y \end{cases} \quad . \tag{4.33}$$

Then

$$(BKs)_j = \sum_{\ell=1}^{r} \sum_{y=1}^{r} \sum_{i=1}^{n} B_{j\ell}\, Z_{\ell y}\, B^T_{yi}\, p_i$$

and (4.4) thus becomes

$$\underline{\dot{s}} = A\underline{s} + BZ(s)B^T\, \underline{p} + \underline{q} \;\; ; \quad \underline{s}(0) = 0 \quad . \tag{4.34}$$

The canonical equations given by (4.32) and (4.34) are thus nonlinear in $\underline{s}$ and $\underline{p}$.

Sufficient Conditions

In order to establish existence of the optimal gain (4.30), conditions 1), 2), 3) and 5) of Appendix B must be satisfied. Condition 1) is implied by the extremal equations (4.30), (4.32) and (4.34). The Legendre condition is obtained from (4.29).

$$\frac{\partial^2 H_3}{\partial K_{ij}\, \partial K_{\ell m}} = \begin{cases} -\,(R_{ij}\, s_j^{\,2} + E_{ij}) & i = \ell,\ j = m \\ 0 & \text{otherwise} \end{cases} \quad . \tag{4.35}$$

Since $E_{ij} > 0 \;\forall i,j$, the extremal is nonsingular and conditions 2) and 3) are always satisfied.

The determination of conjugate points using the methods of Appendix B is, in general, extremely difficult. The second partial derivative matrices in (B.5) are nonlinear in $\underline{s}$ and $\underline{p}$ which makes the computation of the conditions for Theorems B.1 and B.2 a formidable task.

The existence of the optimal gain for the scalar case can, however, be directly proven using Theorem 5 of [43]. With some manipulation, all required hypotheses can be shown to apply. The most difficult is the determination of the constant C for the system and cost inequalities. This can easily be obtained if the term

$$\bar{g} = \sup_{t \in [0,T]} | g(t) |$$

is added to the cost $J(K)$, noting that the minimizing gain will be unaltered. Cesari's Theorem is also applicable to the vector case when $R_{ij} = 0, \forall i,j$. In general $R_{ij} > 0$ for some i,j and then the theorem cannot be applied since the gain and state terms are not functionally separable. It is probable, however, that a slight modification can be made to the theorem to prove existence for the general case.

Solution Techniques

Two methods of solving for the optimal gain matrix (4.30) are given in this section. One is an iterative scheme or gradient technique as described in Appendix B. The other is an approximation method that yields a set of linear equations for which an explicit solution can be obtained.

The gradient method outlined in Appendix B is directly applicable to solving equations (4.4) with $\underline{s}(0) = 0$ and (4.28) subject to the optimality condition (4.29). Equations (4.4) and (4.28) correspond to (B.13) and (B.14) respectively. The elements of (4.29) make up the vector $\underline{H}_K$ used in computing the gain increment (B.17) and the predicted cost error (B.18). The iterative method given by steps a) through d) effectively generates a solution K(t) by forcing (4.29) to approach zero $\forall\, t \in [0,T]$. The rate of convergence is directly effected by the initial choice of K(t) and the step size matrix N. These quantities must be intuitively determined for each problem encountered.

The objective of the approximation method given below is to obtain an explicit, nonsingular solution to the minimum sensitivity problem with state feedback cost described in section 4.2. The method effectively generates an approximate solution to the canonical equations (4.32) and (4.34) when E_{ij} is small. Thus the cost (4.25) will be close to that of (4.5) with the extremal being nonsingular. In addition, it is assumed that the sensitivity terms in the cost have sufficient weight such that the sensitivity vector $\underline{s}(t)$ is small. An approximation to (4.32) and (4.34) can then be obtained by examining the nonlinear elements $\underline{M}$ and Z. From (4.31) the components of $\underline{M}$ are analytic at zero relative to E_{ij} and s_j, i.e.

$$\lim_{E_{ij} \to 0} \left(\lim_{s_j \to 0} M_j \right) = \lim_{s_j \to 0} \left(\lim_{E_{ij} \to 0} M_j \right) = 0 \quad .$$

Thus an expansion of M_j about zero results in

$$\underline{M}\ (\underline{s},\ \underline{p},\ E) \approx 0 \quad .$$

Using this in (4.32) gives

$$\dot{\underline{p}} \approx - A^T \underline{p} + Q\ \underline{s} \quad ; \quad \underline{p}(T) = - D\ \underline{s}(T) \quad . \qquad (4.36)$$

It is seen from (4.33) that the matrix $Z(\underline{s},\ E)$ is not analytic at zero since

$$\lim_{E_{ij} \to 0} \left(\lim_{s_j \to 0} Z(\underline{s},\ E) \right) = 0$$

but

$$\lim_{s_j \to 0} \left(\lim_{E_{ij} \to 0} Z(\underline{s},\ E) \right) = V$$

where V is defined by (4.10). This is to be expected because, at $s_j = 0$, the gain given by (4.8) has a singularity whereas that given by (4.30) is equal to zero. Since $E_{ij} > 0 \ \forall i,j$ and $\underline{s}(0) = 0$, (4.33) indicates that $Z(0, E) = 0$. The sensitivity equation (4.34) therefore initially runs open loop. As the magnitude of $\underline{s}(t)$ increases, the matrix $Z(\underline{s}\ E)$ approaches V. Equation (4.34) will thus be approximated as follows:

$$
\begin{aligned}
\dot{\underline{s}}_1 &= A\,\underline{s}_1 + \underline{g} \quad &;\quad \underline{s}_1(0) = 0 \quad &;\quad 0 < t < T_1 \\
\dot{\underline{s}}_2 &= A\,\underline{s}_2 + BVB^T\,\underline{p} + \underline{g} \quad &;\quad \underline{s}_2(T_1) = \underline{s}_1(T_1) \quad &;\quad T_1 < t < T
\end{aligned}
\tag{4.37}
$$

where $T_1 \in (0, T)$ is a design parameter
and

$$
\underline{s}(t) = \begin{cases} \underline{s}_1(t) & 0 < t < T_1 \\ \underline{s}_2(t) & T_1 < t < T \end{cases} .
$$

Equations (4.36) and (4.37) can be explicitly solved as a coupled system. From (4.30), the optimal gain K(t) is

$$
K_{ij}(t) = \begin{cases} 0 & 0 < t < T_1 \\ \dfrac{s_j}{R_{ij}\, s_j{}^2 + E_{ij}} \cdot \displaystyle\sum_{\ell=1}^{n} p_\ell\, B_{\ell i} & T_1 < t < T \end{cases} . \tag{4.38}
$$

The relationship between the approximate solution given above and that of the singular problem described in sections 4.2 and 4.5 is as follows. The approximation effectively reduces the time interval of optimization and, in doing so, generates an initial sensitivity vector consistent with $\underline{g}(t)$. This was discussed in section 4.5 as a

possible means of satisfying existence conditions. The problem resulting from some components of $\underline{s}(t)$ approaching zero on $(T_1,T]$ still remains, although this in part dictates the choice of T_1. When this occurs, the approximation of W by V on $(T_1,T]$ is no longer valid. The choice of T_1 is further complicated by the fact that the desired trajectory sensitivity may not be attained if T_1 is too large.

In summary, the use of the above approximation and the choice of T_1 is problem dependent and related to the behavior of $\underline{g}(t)$ and to the desired trajectory sensitivity. If the time parameter T_1 can be chosen such that

$$s_j^{\,2}(t) > 0 \quad \forall\, t \in (T_1,T]$$

then equations (4.36) through (4.38) give the desired solution. If not, recourse must be made to the gradient method.

Chapter 5

Comparison Example

5.1 Problem Formulation

It is of interest to compare the reduced and minimum sensitivity solutions discussed in previous chapters. The techniques outlined in Chapter 3 for obtaining the feedback gain function effectively represent the best of the known nondynamic methods presently used for sensitivity reduction. The solution obtained in Chapter 4 minimizes sensitivity relative to (2.34) and (2.37). The question examined in this chapter is how much better does the minimum sensitive (MS) gain perform relative to the regulator (RG) and model following (MF) solutions? A first order example will be described below.

Let the original design system (nominal) be given by

$$\dot{x} = a_n x + b u_n \quad ; \quad x(0) = 10 \tag{5.1}$$

with $u_n = k_n x_n$ obtained from

$$\min_u \frac{1}{2} \int_0^T (x^2 + r u^2) \, dt \tag{5.2}$$

where

$$a_n = 1 \qquad b = 1$$

$$T = 1 \qquad r = .2, 1 \quad .$$

The quantity a_n was chosen positive so that (5.1) as an unforced system would be unstable. This will accentuate the effect of any variation in a_n. Also, two values of r were chosen to vary the form of the nominal trajectory.

The feedback compensator is given by

$$u(t) = u_n(t) + k(t) [y(t) - x_n(t)] \tag{5.3}$$

where u_n and x_n are nominal solutions of (5.1) and (5.2). The actual (real world) system is assumed to be

$$\dot{y} = 1.2y + u(t) \quad ; \quad y(0) = 10 \tag{5.4}$$

where the parameter was varied 20 percent in the unstable direction. Note that if no parameter variations occur, then $y = x_n$ and $u = u_n$. Two measures of the system error are

$$\begin{aligned} \text{Mean Square} &= \int_0^1 (y - x_n)^2 \, dt \\ \text{Final Value} &= | \, y(1) - x_n(1) \, | \quad . \end{aligned} \tag{5.5}$$

The cost of using feedback is measured by

$$\text{Feedback Cost} = \int_0^1 (u - u_n)^2 \, dt \quad . \tag{5.6}$$

Note that if (5.4) is run open loop, then $u = u_n$ and no cost penality is incurred.

The MS compensator is determined as a solution to the following problem

$$\min_K \left[\frac{1}{2} d \, s^2(1) + \frac{1}{2} \int_0^1 (qs^2 + k^2 s^2) \, dt \right] \tag{5.7}$$

subject to

$$\dot{s} = a_n s + bks + x_n(t) \quad ; \quad s(0) = 0 \tag{5.8}$$

which corresponds to that posed in section 2.4. The regulator gain is also obtained from (5.7) and (5.8) but with $x_n(t) = 0$ and $s(0) \neq 0$. For the first order case, the model following control is a constant negative gain such that (5.4) is stabilized.

In this example, the gain k(t) is chosen either as a solution

to (5.7) and (5.8) for various values of d and q or as a variable negative constant $k(t) = - k_o$. This determines the feedback control (5.3) to be used in the actual system (5.4). Equation (5.4) is then integrated to obtain the performance measures (5.5) and (5.6). A comparison of the feedback cost required to obtain a given sensitivity reduction can thus be obtained for the MS, RG and MF compensators by varying d, q and k_o.

5.2 Numerical Results

The problem posed in the previous section was programmed on a digital computer with Runge-Kutta techniques used to integrate (5.4), (5.5) and (5.6). The major results are shown in Figures 5.1 through 5.4. In each case an open loop trajectory was generated with $k(t) \equiv 0$. A suitable goal for error reduction with feedback was then taken as 10 percent of the open loop error.

Figures 5.1 and 5.2 present the comparison results for a decreasing nominal trajectory, $x_n(t)$, obtained with r = .2 in (5.2). It is seen that the MS and MF compensators respectively give the lowest and highest errors for equivalent feedback costs. To achieve ten percent of the open loop mean squared error, Figure 5.1 indicates that the MS gain requires 30 percent less feedback effort than does the regulator. The reduction of the final error is not as great in Figure 5.2, however, the final value sensitivity cost term, d, in (5.7) was zero for those runs.

Similar results are depicted in Figures 5.3 and 5.4 for an approximately constant nonimal trajectory obtained with r = 1 in (5.2). To achieve 10 percent of the open loop mean squared error, the MS gain requires 50 percent less feedback effort than does the regulator. The MS gain did not always exist, as was discussed in section 4.5. This situation was remedied by employing a nonzero initial sensitivity term. The results are shown in the figures which indicate that the MS gain still gives a significant performance improvement over that of the regulator.

5.3 Discussion

Some observations can be made concerning the MS and regulator gains which are both time varying. It appears that the existence of the MS gain is determined in part by the stability of the nominal trajectory. Some cases were run with an increasing nominal trajectory which resulted in greater existence problems than those of Figures 5.3 and 5.4. Also the inclusion of small final value constraints ($d > 0$ in (5.7)) for the MS and regulator gains gave no significant relative performance change. In addition. runs were made with a stable nominal trajectory ($a_n = -1$ in equation (5.1)). Less performance difference between the MS and regulator occurred than with the unstable nominal. This is to be expected since stable systems automatically reduce the effects of disturbances.

One possible drawback in using the MS gain over the regulator is that the former is more susceptible to measurement errors. Since the MS gain must initially be bounded, it will in general have a larger average magnitude over the interval than will the regulator. However, if the parameter variations are of significant magnitude and adequate prefiltering is done, measurement noise should produce negligable effects.

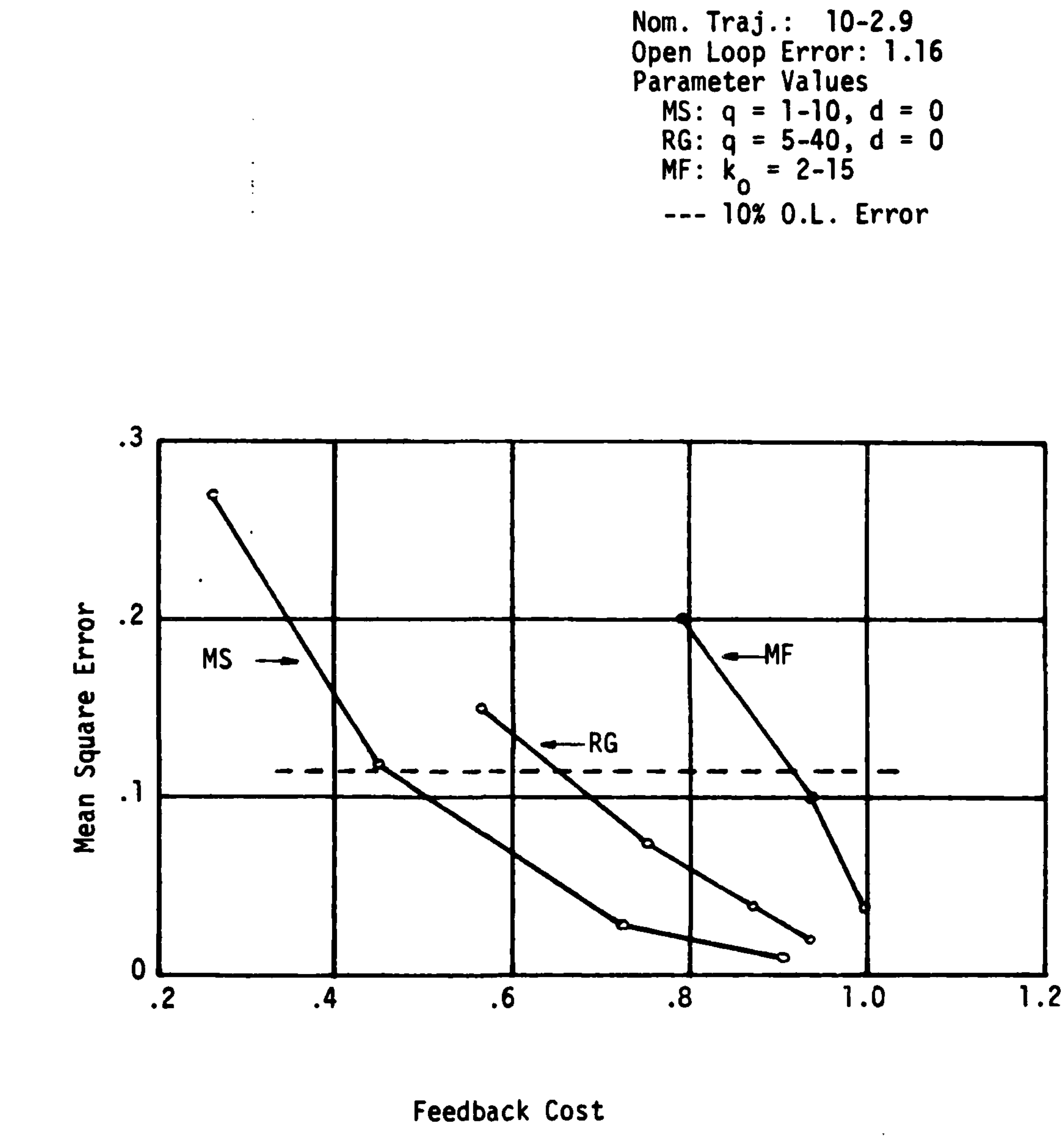

Figure 5.1 Mean Square Error
vs
Feedback Cost for Decreasing Nominal Trajectory

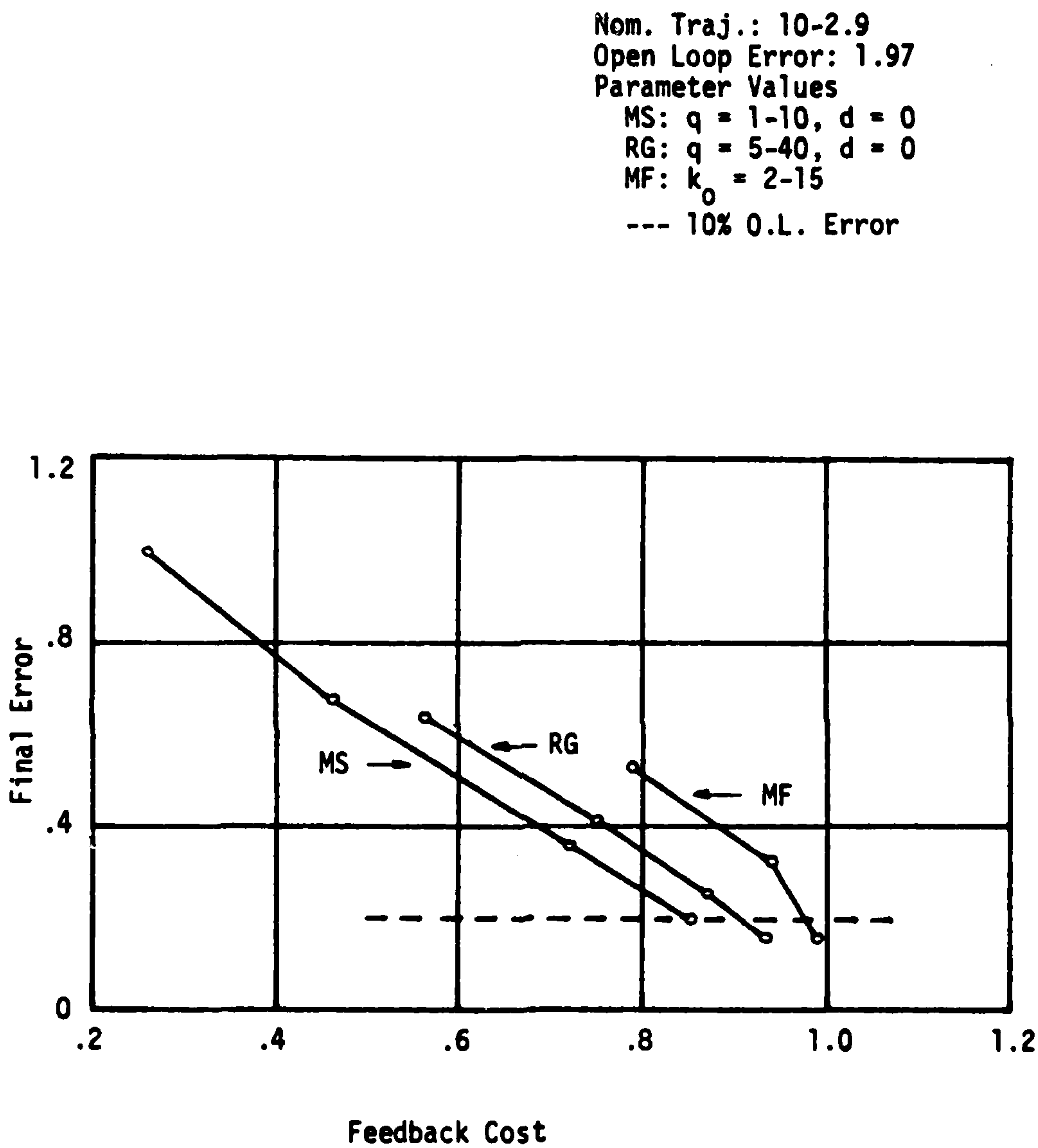

Figure 5.2: Final Error
vs
Feedback Cost for Decreasing Nominal Trajectory

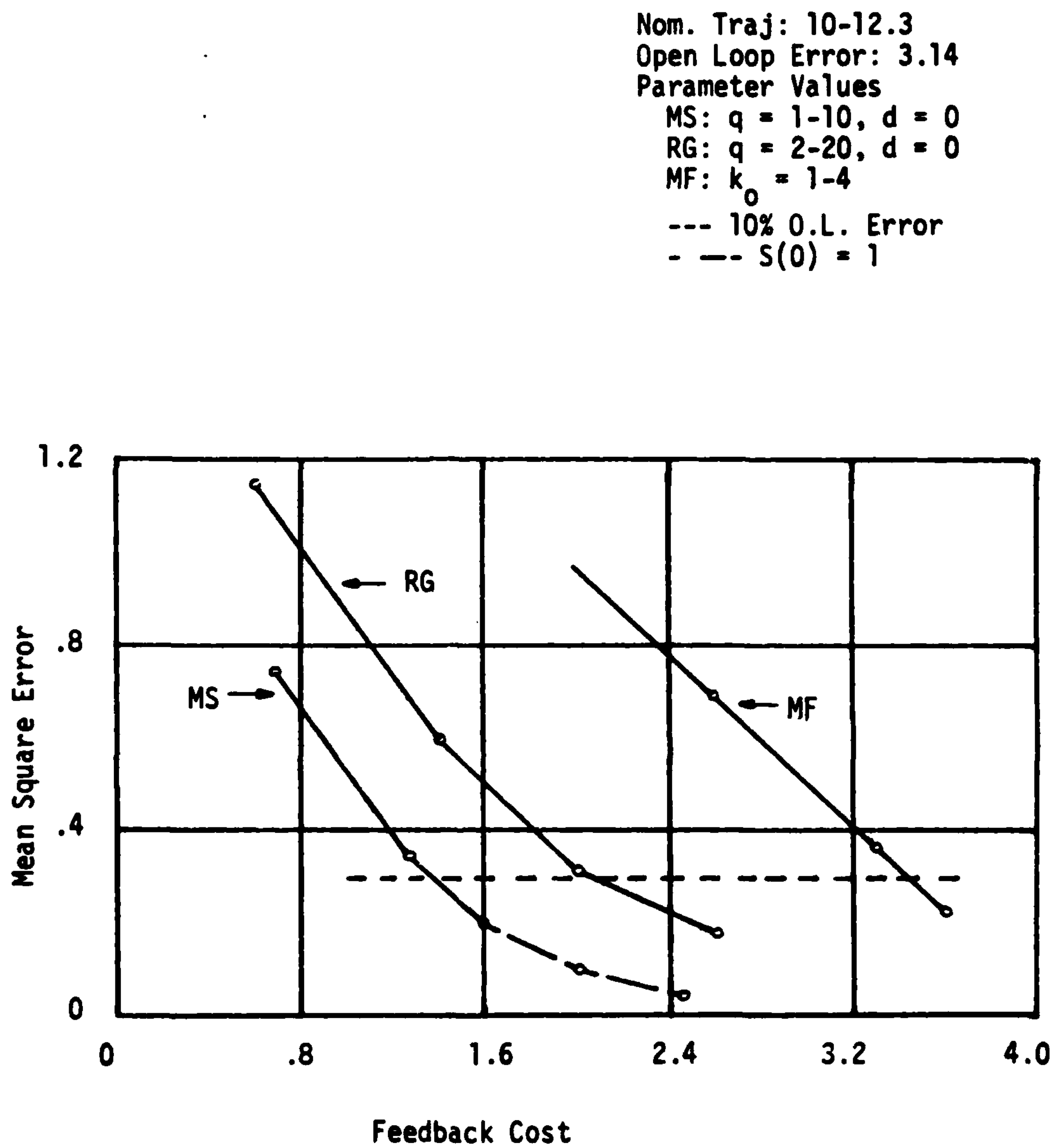

Figure 5.3: Mean Square Error
vs
Feedback Cost for Approximately Constant Nominal Trajectory

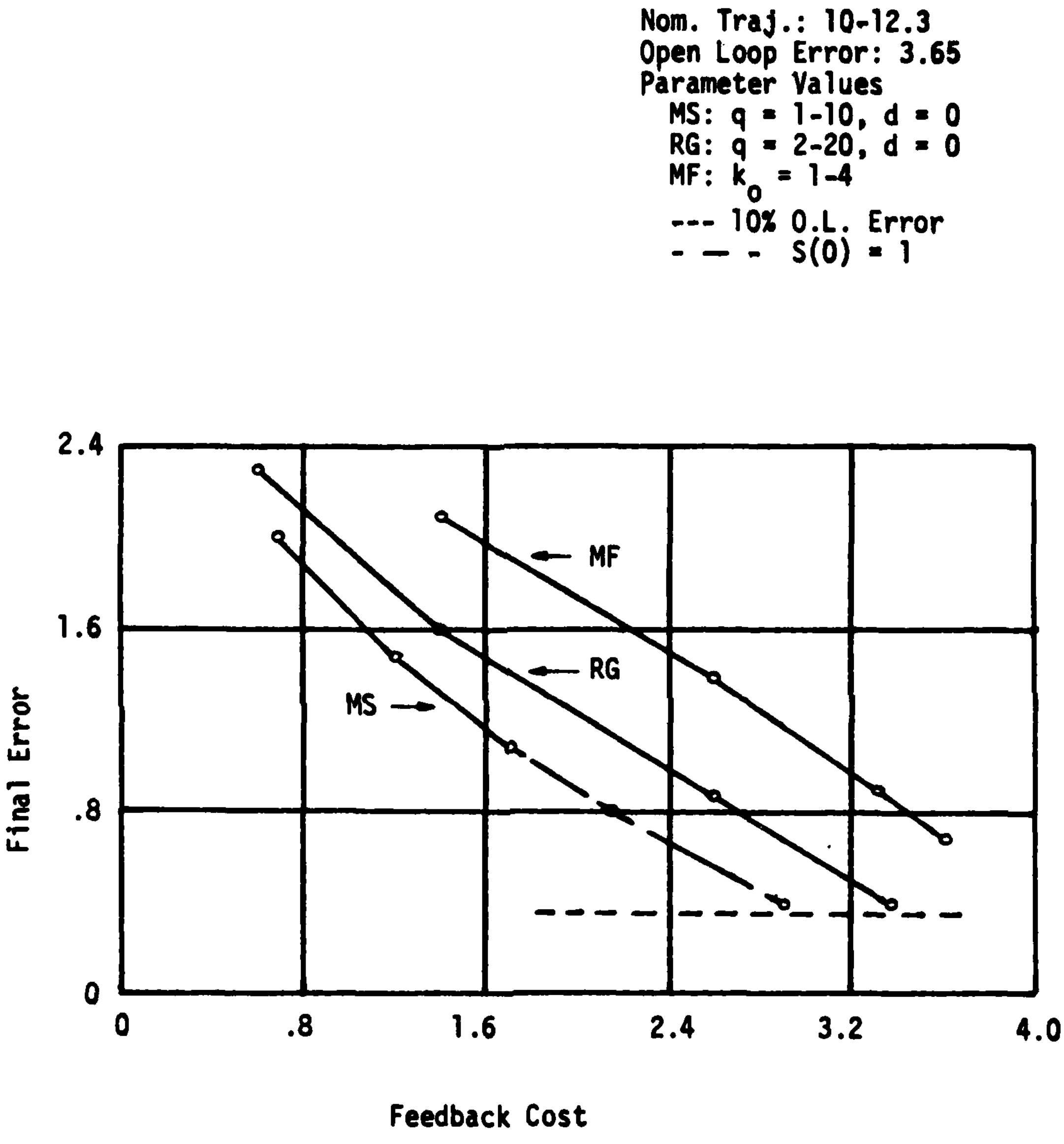

Figure 5.4: Final Error
vs
Feedback Cost for Approximately Constant Nominal Trajectory

Chapter 6

Conclusions

6.1 Summary

A major contribution of this dissertation is the formulation of the trajectory sensitivity problem as a direction field problem in the calculus of variations. This formulation is unique in that it is applicable to a large class of nonlinear systems which previously could not be handled by standard sensitivity methods. It also has the desirable property of reducing, under certain conditions, to the classical formulation of the sensitivity problem.

The principal result obtained from the new formulation is the development of a theory for the practical design of linear feedback compensators which minimize trajectory sensitivity. With the assumptions of small parameter variations and quadratic sensitivity cost terms, the general problem reduces to one for which an explicit noniterative solution can be obtained for a linear feedback gain function. The necessary and sufficient conditions developed for the minimum sensitive gain effectively extend the regulator theory developed by Kalman to include unknown constant disturbances.

6.2 Extensions

There are at least three ways in which the results of this dissertation can be extended. The first and probably the most fruitful is to generate the linear feedback controls in section 2.4 from dynamic systems. The possible advantages of this over pure gain feedback are that the singularity problems could be removed by including rate limiting terms in the cost and that the sensitivity to measurement disturbances might be decreased. Also, in view of the advantages of the minimum sensitive gain control over least square parameter estimators, a comparison between dynamic minimum sensitive controllers and dynamic parameter estimators (Kalman filter or observer) could yield equivalent results.

The direction field formulation of the sensitivity problem can be extended to include system functions which exhibit convergence in the mean on the parameter set instead of being continuously differentable relative to the parameter vector. This will require a slight modification to Gamkrelidze's proof of the necessary conditions for an extremal.

The first order sensitivity problem derived in section 2.4 could be generalized to include higher order sensitivity terms by relaxing the assumption of small parameter variations. In fact, the problem could be formulated in an infinite dimensional space to minimize all orders of trajectory sensitivity. This would cause the general solution to be completely independent of parameter errors.

References

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Appendix A

Functional Analysis Formulation of the Trajectory Sensitivity Problem

This appendix explicitly relates the ideas and techniques developed in Chapters 2 and 4 to classical sensitivity methods for linear systems discussed in Chapter 1. The sensitivity reduction and disturbance rejection characteristics of forward loop and input compensators are shown to be equivalent. It is also demonstrated that the linear feedback compensator of section 2.4 is similar in structure to classical input compensators. The most interesting result obtained is the correspondence between the measurement noise transfer function and the closed loop system error introduced in Chapter 2 to limit the amount of applied feedback.

Let the closed loop system be represented by

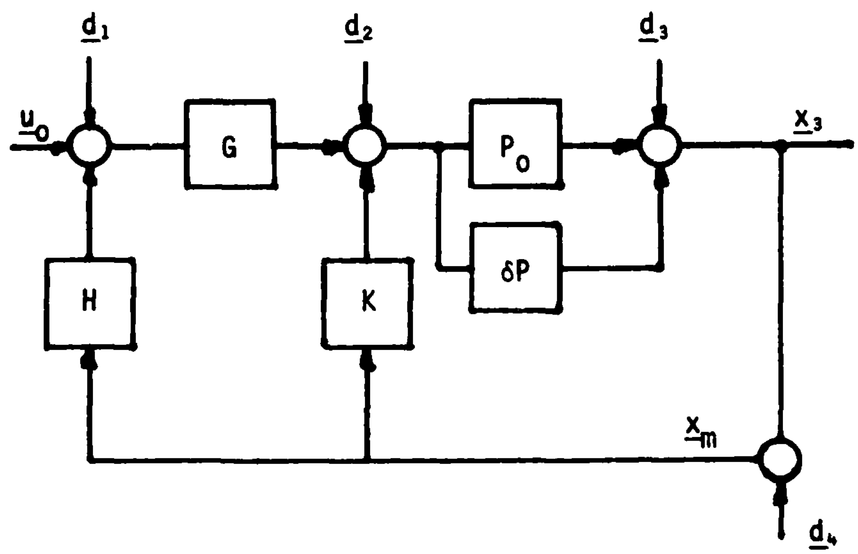

Figure A.1: Combined Closed Loop System

where the linear operators P_0, G, K and H are defined on appropriate Banach spaces, the $\underline{d}_i$, i = 1,...,4 represent unknown disturbances and δP is an additive linear variation in the nominal plant. The desired or nominal output is given by

$$\underline{x}_{3n} = P_o \, \underline{u}_o \quad . \tag{A.1}$$

The open loop system, H = K = 0 and G = I, yields

$$\underline{x}_{30} = \underline{d}_3 + P\underline{d}_2 + P\underline{d}_1 + P\underline{u}_o \tag{A.2}$$

where $P = P_0 + \delta P$ is the actual plant operator. If the sensitivity operator is defined as

$$N = [\ I - PGH - PK\ ]^{-1} \tag{A.3}$$

then the output of the closed loop system is

$$\underline{x}_{3c} = N\ [\ \underline{d}_3 + P\underline{d}_2 + PG\underline{d}_1 + PGH\underline{d}_4 + PK\underline{d}_4 + PG\underline{u}_o\ ]\ . \tag{A.4}$$

To make the above nominally equivalent relative to (A.1) when $\underline{d}_i = 0$, i = 1,...,4, and $P = P_0$, the compensator must be

$$G = [\ I - K\,P_0\ ]\,[\ I + H\,P_0\ ]^{-1}\ . \tag{A.5}$$

Let the closed loop error be given by $\delta\underline{x}_c = \underline{x}_{3c} - \underline{x}_{3n}$ and similarily the open loop error by $\delta\underline{x}_o = \underline{x}_{30} - \underline{x}_{3n}$. Then from (A.1) through (A.5) the following holds

$$\underline{\delta x}_c = N\ \ \delta\underline{x}_o + [\ N - I\ ]\ (\underline{d}_4 - P_o\underline{d}_1)\ . \tag{A.6}$$

The above indicates that when feedback is applied to make N a contradiction operator (approach zero) and thus reduce the effects of the open loop system errors, a corresponding increase occurs in the output error due to measurement noise. This result is similar to that

obtained in Chapter 3. Equation (A.6) also shows that input compensation (H = 0) and forward loop compensation (K = 0) have identical error rejection characteristics.

When input compensation is employed, the plant input is

$$\underline{u}_p = G\,\underline{u}_o + K\,\underline{x}_3$$

where all disturbances are assumed zero. Using (A.5) the above becomes

$$\underline{u}_p = [\,I - K\,P_o\,]\,\underline{u}_o + K\,\underline{x}_3$$

or from (A.1)

$$\underline{u}_p = \underline{u}_o + K\,[\,\underline{x}_3 - \underline{x}_{3n}\,] \tag{A.7}$$

which is similar in form to the MS compensator of Chapter 4.

The transfer function for measurement noise is obtained from (A.4) as

$$T_M = N\,P\,[\,G\,H + K\,] \quad . \tag{A.8}$$

The closed loop system error, ΔS, as defined in Chapter 2 is given by the difference between the closed loop and open loop transfer functions when only parameter variations occur. Thus from (A.2) and (A.4)

$$\Delta S = N\,P\,G - P \quad . \tag{A.9}$$

Using (A.3) and (A.5) the above reduces to

$$\Delta S = N\,P\,[\,G\,H + K\,]\,\delta P \quad . \tag{A.10}$$

It is therefore seen that the closed loop system error (A.10), normalized with respect to the parameter variations, is identical to the transfer function (A.8) for measurement noise.

Appendix B
Existence and Solution Techniques for Variational Problems

B.1 Necessary and Sufficient Conditions for a Relative Minimum

Sufficient conditions are determined in this section which are mainly applicable to the problems considered in Chapter 4. Some new results are derived for the existence of conjugate points and their relationship to controllability. The variational problem considered is stated as follows:

$$\min_{K} J(T, \underline{s}, K) = \psi(T, \underline{s}) + \int_{0}^{T} L(t, \underline{s}, K)\, dt \qquad \text{(B.1)}$$

subject to

$$\dot{\underline{s}} = \underline{f}(t, \underline{s}, K) \quad , \quad \underline{s}(0) = \underline{s}_0 \qquad \text{(B.2)}$$

where the functions ψ, L and $\underline{f}$ are continuous in t and C^2 WRT $\underline{s}$ and K, $\underline{s}$ is an n dimensional vector and K an (rxn) array. It is assumed that the endpoint of $\underline{s}$ is free, the time intertval T is fixed and K is unconstrained. Defining the Hamiltonian as

$$H(t, \underline{s}, \underline{p}, K) = - L(t, \underline{s}, K) + \underline{p}^T \underline{f}(t, \underline{s}, K)$$

where $\underline{p}(t)$ is an n vector, the following conditions can be defined concerning the solution of (B.1) and (B.2).

1) There exists an absolutely continuous vector $\underline{p}(t)$ which satisfies

$$\dot{\underline{p}} = -[\partial H/\partial s_1 \ldots \partial H/\partial s_n]^T \triangleq - \nabla_s H \quad ; \underline{p}(T) = -\nabla_s \psi|_{t=T}$$

with

$$\dot{\underline{s}} = \nabla_p H \quad ; \quad \underline{s}(0) = \underline{s}_0$$

and where K is determined from

$$\frac{\partial H}{\partial K_{ij}} \triangleq H_{K_{ij}} = 0 \ , \ \forall \ i,j \ .$$

A solution of the above equations ($\underline{s}_n(t)$, $\underline{p}_n(t)$, $K_n(t)$) is called an extremal.

2) The extremal is nonsingular, i.e.,

$$\text{DET} \left[\frac{\partial^2 H}{\partial K_{ij} \, \partial K_{\ell m}} \right] \neq 0 \quad .$$

3) Legendre - the matrix

$$\left[\frac{\partial^2 H}{\partial K_{ij} \, \partial K_{\ell m}} \right] \quad .$$

is negative definite along the extremal.

4) Weierstrass - The function

$$E(t, \underline{s}_n, K_n, K) = H(t, \underline{s}_n, K_n, \underline{p}_n) - H(t, \underline{s}_n, K, \underline{p}_n)$$

is such that $E > 0$ along the extremal $\forall$ K.

5) No conjugate points exist on the half closed interval $(0, T]$.

From references [30] and [40], necessary conditions for a relative minimum are 1), 3), 4) and 5) with 3) and 5) relaxed as follows. The Legendre condition requires only negative semi-definiteness and conjugate points need be considered only on the open interval.

Sufficient conditions for a relative minimum in $(t, \underline{s}, \dot{\underline{s}})$ space (weak min) are 1), 2), 3) and 5). For a strong minimum in $(t, \underline{s})$ space, these conditions must be strengthened by requiring 3) or 4) to hold in some neighborhood of the extremal. In what follows, only weak minima will be considered. Since the existence of conjugate points is somewhat difficult to determine, the next few paragraphs are included to give a simple explaination of them along with necessary and sufficient conditions for their existence.

As in ordinary calculus, the existence of a minimum is directly related to the positiveness of the second derivative or variation of the function. The second variation of (B.1) subject to variations δK about the extremal determined from 1) is

$$\partial^2 J = \frac{1}{2}\,\delta \underline{s}^T(T)\,[\partial^2\psi/\partial s^2]\,\delta\underline{s}(T) + \frac{1}{2}\int_0^T [\delta\underline{s}\ \delta K]\begin{bmatrix} H_{ss} & H_{sK} \\ H_{Ks} & H_{KK}\end{bmatrix}\begin{bmatrix}\delta\underline{s} \\ \delta_K\end{bmatrix} dt \tag{B.3}$$

where $\delta\underline{s}(t)$ is obtained from

$$\delta\dot{\underline{s}} = \frac{\partial f}{\partial s}\,\delta\underline{s} + \frac{\partial f}{\partial K}\,\delta K \quad ; \quad \delta\underline{s}(0) = 0 \tag{B.4}$$

and all partials and variations are defined relative to the extremal. One method of showing that $\delta^2 J > 0 \;\forall\; \delta K \neq 0$ is to prove that $\delta^2 J$ does not have a minimum WRT δK. To this end, the necessary conditions for a minimum of (B.3) subject to (B.4) yield the canonical equations

$$\begin{bmatrix}\delta\dot{\underline{s}} \\ \delta\dot{\underline{p}}\end{bmatrix} = \begin{bmatrix}\overline{H}_{ps} & \overline{H}_{pp} \\ -\overline{H}_{ss} & -\overline{H}_{sp}\end{bmatrix}\begin{bmatrix}\delta\underline{s} \\ \delta\underline{p}\end{bmatrix} \tag{B.5}$$

$$\delta\underline{s}(0) = 0 \quad ; \quad \delta\underline{p}(T) = -D\ \ \delta\underline{s}(T) \tag{B.6}$$

where $\delta\underline{p}(t)$ is an n vector and

$$\overline{H}(t,\underline{s},\underline{p}) = \max_K H(t,\underline{s},\underline{p},K)$$

$$D = \partial^2\psi/\partial s^2 \quad .$$

If it can be shown that (B.5) subject to (B.6) does not have a solution on [0, T], then the necessary conditions are not satisfied and $\delta^2 J > 0$. Thus consider variations $\delta\underline{s}$ and $\delta\underline{p}$ which satisfy only the terminal condition of (B.6). A conjugate point will be defined relative to the

final time as in [41], instead of the initial time, so that terminal conditions can be more adequately accounted for.

Definition: A conjugate point occurs at $t_c \in [0, T)$ if $\exists$ a nontrivial solution $(\delta\underline{s}, \delta\underline{p})$ to (B.5) on $(t_c, T]$ satisfying $\underline{p}(T) = -D\, \delta\underline{s}(T)$ such that $\delta\underline{s}(t_c) = 0$.

If a conjugate point exists, then the nonzero variation,

$$\delta\underline{s}'(t), \delta\underline{p}'(t) = \begin{cases} \delta\underline{s}(t), \delta\underline{p}(t) & t_c < t < T \\ 0 \quad , \delta\underline{p}_0(t) & 0 < t < t_c \end{cases}$$

where $\delta\dot{\underline{p}}_0 = -\bar{H}_{sp}\, \delta\underline{p}_0 \quad ; \quad \delta\underline{p}_0(t_c) = \delta\underline{p}(t_c)$

is a solution to (B.5) subject to (B.6).

The following theorem, which evolved from a definition of conjugate points in [40] and [41], gives explicit conditions for their nonexistence.

Theorem B.1: A necessary and sufficient condition for no conjugate points to exist on $[0,T)$ relative to the problem defined by (B.1) and (B.2) is that the $(n\times n)$ matrix $\theta(t, T)$ be nonsingular on $[0, T)$ where

$$\theta(t,T) = [\, \psi_{11}(t, T) - \psi_{12}(t, T)\, D \,] \tag{B.7}$$

and

$$\begin{bmatrix} \psi_{11}(t, T) & \psi_{12}(t, T) \\ \psi_{21}(t, T) & \psi_{22}(t, T) \end{bmatrix} \tag{B.8}$$

is the $(2n \times 2n)$ transition matrix of (B.5) with D defined by (B.6).

Proof: Using (B.8) the solution of (B.5) can be represented as

$$\delta \underline{s}(t) = \psi_{11}(t, T)\ \delta \underline{s}(T) + \psi_{12}(t, T)\ \delta \underline{p}(T)$$
$$\delta \underline{p}(t) = \psi_{21}(t, T)\ \delta \underline{s}(T) + \psi_{22}(t, T)\ \delta \underline{p}(T) \quad . \tag{B.9}$$

Then with (B.6) the above becomes

$$\delta \underline{s}(t) = \theta(t, T)\ \delta \underline{s}(T)$$
$$\delta \underline{p}(t) = [\ \psi_{21}(t, T) - \psi_{22}(t, T)\ D\ ]\ \delta \underline{s}(T) \quad . \tag{B.10}$$

It is therefore seen that any nontrival variation must result from a nonzero $\delta \underline{s}(T)$. To prove sufficiency, assume that $\exists$ a conjugate point at $t_c \in [0, T)$. Thus from (B.10)

$$0 = \theta(t_c, T)\ \delta \underline{s}(T)$$

which, since $\theta(t_c, T)$ is nonsingular, implies that $\delta \underline{s}(T) = 0$, a contridiction. To prove necessity assume that $\exists$ a $t' \in [0, T)$ st $\theta(t', T)$ is singular. Then $\exists$ a nonzero vector $\underline{\beta}$ st

$$0 = \theta(t', T)\ \underline{\beta}\ .$$

Defining $\delta \underline{s}(T) = \underline{\beta}$, the continuity of $\theta(t, T)$ and the fact that $\theta(T, T) = I$ imply that $\delta \underline{s}(t) \neq 0$ for some $t \in (t', T)$ near T. Therefore, by definition, a conjugate point occurs at t'. This contridiction thus completes the proof.

Since the transition matrix for time varying systems is in general difficult to obtain, its determination can be bypassed with the following.

Theorem B.2: If the matrices $\bar{H}_{pp}$, $-\bar{H}_{ss}$ and D defined by (B.5) and (B.6) are positive semidefinite on [0, T], then no conjugate points exist on [0,T).

Proof: Assume that $\exists$ a $t' \in [0, T)$ such that $\theta(t', T)$ is not

invertable. Then $\exists$ a nonzero constant vector $\underline{\beta}$ st

$$0 = \theta(t', T)\,\underline{\beta}\ .$$

With some manipulation, the following equation can be obtained from (B.5)

$$\delta\underline{p}^T(t)\ \delta\underline{s}(t) = \delta\underline{p}^T(T)\ \delta\underline{s}(T) + \int_T^t (\delta\underline{p}^T\ \bar{H}_{pp}\delta\underline{p} - \delta\underline{s}^T\ \bar{H}_{ss}\ \delta\underline{s})\ d\tau\ .$$

Let $\delta\underline{s}(T) = \underline{\beta}$ and $t = t'$, then from (B.6) and (B.10)

$$0 = -\ \delta\underline{s}^T(T)\ D\ \delta\underline{s}(T) - \int_{t'}^T (\delta\underline{p}^T\ \bar{H}_{pp}\ \delta\underline{p} + \delta\underline{s}^T\ (-\bar{H}_{ss})\ \ \delta\underline{s})\ d\tau\ .$$

Since D and $-\bar{H}_{ss}$ are positive semidefinite, the above implies that

$$-\int_{t'}^T (\delta\underline{p}^T\ \bar{H}_{pp}\ \delta\underline{p})\ d\tau \geq 0\ .$$

But then

$$\bar{H}_{pp}\ \delta\underline{p} = 0 \qquad \forall\ t \in [t', T]$$

because of the positive semidefiniteness and continuity of $\bar{H}_{pp}$. The $\delta\dot{\underline{s}}$ term in (B.5) is thus uncoupled from $\delta\underline{p}$ and

$$\delta\underline{s}(t) = \phi(t, T)\ \ \delta\underline{s}(T)$$

holds $\forall\ t \in [t', T]$ where $\phi(t, T)$ is the transition matrix for

$$\delta\dot{\underline{s}} = \bar{H}_{ps}\ \delta\underline{s}\ .$$

At $t = t'$, $\delta\underline{s}(t') = 0$ which by the invertability of $\phi(t', T)$ implies that $\delta\underline{s}(T) = 0$, a contridiction. Therefore $\theta(t', T)$ must be invertable and by Theorem B.1 the proof is complete.

Since the minimization problem defined by (B.1) and (B.2) has

variable endpoints $\underline{s}(T)$ and fixed time interval [0, T], controllability as introduced in [11] does not play an important role in the conditions for an extremal. However, if the endpoints of (B.2) are fixed ($\underline{s}(T) = \underline{s}_2$), then there is a direct relationship between controllability and conjugate points as shown below.

With fixed endpoints, the canonical equations for the accessory minimization problem are given by (B.5) with (B.6) replaced by

$$\delta\underline{s}(0) = 0 \quad ; \quad \delta\underline{s}(T) = 0 \quad . \tag{B.11}$$

Conjugate points can now be defined relative to the initial time t = 0, which is the classical definition [30].

<u>Theorem B.3</u>: If the (nxn) matrix $\psi_{12}(t, 0)$ given by (B.8) with T = 0 is nonsingular, then no conjugate points exist on (0, T] relative to the problem defined by (B.1) and (B.2) with $\underline{s}(T)$ fixed.

Proof: This is similar to the first part of the proof of Theorem B.1 and will therefore not be repeated.

<u>Theorem B.4</u>: If the matrices $\overline{H}_{pp}$ and $-\overline{H}_{ss}$ of (B.5) are positive semidefinite on (0, T] and if (B.5) is completely controllable, i.e. the matrix

$$W(0, t) = \int_0^t \phi(0, \tau)\ \overline{H}_{pp}(\tau)\ \phi^T(0, \tau)\ d\tau$$

is positive definite $\forall\ t \in (0, T]$ with $\phi(\cdot)$ the transition matrix for $\overline{H}_{ps}$, then no conjugate points exist on (0, T] for the fixed endpoint problem.

Proof: Using Theorem B.3, the proof reduces to showing that if $W(0, t)$ is positive definite then $\psi_{12}(t, 0)$ is nonsingular. This was shown in reference [42] which contains existence theorems for solutions of linear two point boundary value problems.

It should be noted that when (B.2) is linear and (B.1) quadratic, such as (4.4) and (4.5), the controllability of (B.5) is implied by the feedback controllability of (B.2).

B.2 A First Order Gradient Method for the Solution of Nonlinear Optimization Problems

The gradient method described below is a standard first order technique for solving nonlinear problems (reference [44]). The method is simple to apply although convergence in some cases can be extremely slow. It is, however, quite applicable to the unconstrained minimization problem defined in section B.1.

Adjoining the sensitivity equation (B.2) to the cost (B.1) with multipliers $\underline{p}(t)$ results in

$$J = \psi(T, \underline{s}) + \int_0^T [L(t, \underline{s}, K) + \underline{p}^T (\dot{\underline{s}} - \underline{f})] \, dt \quad .$$

Using the Hamiltonian defined in section (B.1), the above becomes

$$J = \psi(T, \underline{s}) + \int_0^T [-H (t, \underline{s}, \underline{p}, K) + \underline{p}^T \dot{\underline{s}}] \, dt$$

or integrating by parts

$$J = \psi(T, \underline{s}) + \underline{p}^T \underline{s} \Big|_0^T - \int_0^T [H(t, \underline{s}, \underline{p}, K) + \dot{\underline{p}}^T \underline{s}] \, dt \ . \tag{B.12}$$

The gradient method basically consists of choosing an initial value for the optimal gain $K_1(t)$ and then using it to integrate the canonical equations

$$\dot{\underline{s}} = \nabla_p H \quad , \quad \underline{s}(0) = \underline{s}_0 \tag{B.13}$$

$$\dot{\underline{p}} = - \nabla_s H \ , \quad \underline{p}(T) = - \nabla_s \psi \quad . \tag{B.14}$$

A new gain $K_2(t)$ is determined to minimize the predicted error in the cost (B.12). The process is then repeated until the cost error is near zero.

The perturbation of (B.12) relative to $\delta \underline{s}$ and δK is

$$\delta J = [\nabla_s^T \psi]\delta \underline{s}(T) + \underline{p}^T \delta \underline{s}(T)$$

$$- \int_0^T [[\nabla_s^T H]\delta \underline{s} + \sum_{i=1}^{r} \sum_{j=1}^{n} \frac{\partial H}{\partial K_{ij}} \delta K_{ij} + \dot{\underline{p}}^T \delta \underline{s}] \, dt \, .$$

Using (B.14) to define $\underline{p}(t)$, the above becomes

$$\delta J = - \int_0^T [\sum_{i=1}^{r} \sum_{j=1}^{n} \frac{\partial H}{\partial K_{ij}} \delta K_{ij}] \, dt$$

or

$$\delta J = - \int_0^T \underline{H}_K^T \delta \underline{K} \, dt \tag{B.15}$$

where $\underline{H}_K$ and $\delta \underline{K}$ are $r \cdot n$ dimensional vectors with elements $\frac{\partial H}{\partial K_{ij}}$ and δK_{ij} respectively. In order to limit the size of δK, a quadratic term is added to (B.15) as follows:

$$\delta J_T = - \int_0^T \underline{H}_K^T \delta \underline{K} \, dt - \frac{1}{2} \int_0^T \delta \underline{K}^T N \delta \underline{K} \, dt \tag{B.16}$$

where N is a symmetric, positive definite, $(n \cdot r \times n \cdot r)$ dimensional matrix. A necessary condition for a minimum of δJ_T is

$$\underline{H}_K + N \delta \underline{K} = 0$$

or

$$\delta \underline{K} = - N^{-1} \underline{H}_K \quad . \tag{B.17}$$

Using (B.17) to compute the new gain as

$$K_{ij} : K_{ij} + \delta K_{ij} \quad .$$

The cost deviation is

$$\delta J = \frac{1}{2} \int_0^T \underline{H}_K^T N^{-1} \underline{H}_K \, dt \quad . \tag{B.18}$$

The gradient method of computing the optimal gain is thus composed of the following steps:

a) Choose an initial gain $K(t)$ and step size matrix N.

b) Integrate (B.13) forward from $\underline{s}_o$ and then (B.14) backwards to obtain $\underline{s}(t)$ and $\underline{p}(t)$ on $[0, T]$.

c) Determine the new gain from (B.17) and the predicted cost error from (B.18).

d) Repeat the above steps until the predicted cost error is approximately zero.

APPENDIX C

Least Square Parameter Estimation and Minimum Sensitive Control

C.1 Introduction

As stated in Chapter 1, one solution to the sensitivity problem is to first estimate the values of the unknown parameters and then use the estimates to control the system output near the nominal. The standard technique is to include the parameters as part of the state and to obtain the augmented state estimates using a Kalman filter. Then the regulator solution can be used for feedback control (section 1.2.5). If it is assumed that initial condition errors are included in the nominal control and that the total state, $\underline{x}(t)$, can be measured, a nondynamic least square estimate of the parameters can be obtained from $\underline{x}(t)$. The combined estimator and controller is thus given by a linear gain function.

In this appendix the feedback control obtained with the least square parameter estimator will be compared to the minimum sensitive gain function derived in Chapter 4. It will be shown that both feedback gains have similar structures when the number of uncertain parameters is equal to the dimension of the state, i.e. $m = n$. However, when $m < n$ the least square (LS) gain must be determined by a nonlinear set of differential equations with time varying coefficients, and when $m > n$ the LS gain doesn't exist. Neither of these problems occur with the MS gain. Other advantages which result from employing the MS control will also be pointed out.

C.2 Least Square Controller

Assuming that the closed loop system dynamics are described by (4.1), the linear perturbation equation is given by

$$\underline{\Delta\dot{x}} = \frac{\partial f}{\partial x}\,\underline{\Delta x} + \frac{\partial f}{\partial u}\,\underline{\Delta u} + \frac{\partial f}{\partial \eta}\,\underline{\Delta\eta} \quad ; \quad \underline{\Delta x}\,(0) = 0 \qquad \text{(C.1)}$$

where all partials are evaluated along the nominal and the Δ quantities represent off nominal errors. Any initial condition error is assumed to be included in $\underline{u}_{\eta}(t)$. If the parameter influence matrix is defined by

$$G = [\ \frac{\partial f}{\partial \eta}\ ] = [\ \underline{g}_1, \ldots, \underline{g}_m\ ]\ ,$$

then the above equation becomes,

$$\underline{\Delta \dot{x}} = A\ \underline{\Delta x} + B\ \underline{\Delta u} + G\ \underline{\Delta \eta}\ \ ;\ \underline{\Delta x}\ (0) = 0\ . \qquad (C.2)$$

Since it is desired to reduce the trajectory error, $\underline{\Delta x}(t)$, the following cost functional will be employed

$$J_L = \frac{1}{2}\ \underline{\Delta x}^T(T) D\ \underline{\Delta x}(T) + \frac{1}{2} \int_0^T (\ \underline{\Delta x}^T Q\ \underline{\Delta x} + \underline{\Delta u}^T R\ \underline{\Delta u}\ ) d\tau \quad (C.3)$$

where D, Q and R are as defined in Chapter 4. The control error term is included in the cost to limit the amount of applied feedback. The minimization of (C.3) subject to (C.2) results from [11] in the following control

$$\underline{\Delta u} = R^{-1} B^T P_1\ \underline{\Delta x} + R^{-1} B^T P_2\ \underline{\Delta \eta} \qquad (C.4)$$

where the (nxn) and (nxm) matrices P_1 and P_2 satisfy

$$-\dot{P}_1 = A^T P_1 + P_1 A + P_1 B R^{-1} B^T P_1 - Q\ \ ;\ \ P_1(T) = -D \qquad (C.5)$$

$$-\dot{P}_2 = A^T P_2 + P_1 G + P_1 B R^{-1} B^T P_2 \qquad ;\ \ P_2(T) = 0\ . \qquad (C.6)$$

The zero initial condition on the system equation allows an estimate of $\underline{\Delta \eta}$ to be obtained as follows. Using (C.4) in (C.2) and defining $H = BR^{-1}B^T$, the closed loop system error is described by

$$\underline{\Delta \dot{x}} = (A + HP_1)\ \underline{\Delta x} + (G + HP_2)\ \underline{\Delta \eta} \quad ; \quad \underline{\Delta x}(0) = 0 \qquad (C.7)$$

or

$$\underline{\Delta x}(t) = \int_0^T \psi(t,\tau)\ [\ G + HP_2\ ]\ \underline{\Delta \eta}\ d\tau$$

where $\psi(\cdot)$ is the transition matrix for the free part ($\underline{\Delta \eta} = 0$) of (C.7). Since $\underline{\Delta \eta}$ is assumed to be constant

$$\underline{\Delta x}(t) = W(t)\ \underline{\Delta \eta} \qquad (C.8)$$

where

$$W(t) = \int_0^T \psi(t,\tau)\ [\ G + HP_2\ ]\ d\tau \quad .$$

Differentiating the above results in

$$\dot{W} = [\ A + HP_1\ ]\ W + [\ G + HP_2\ ] \quad ; \quad W(0) = 0 \qquad (C.9)$$

Assuming that the state error can be measured as

$$\underline{\Delta x}(t) = W(t)\ \underline{\Delta \eta} + \underline{d}\ ,$$

where $\underline{d}$ represents zero mean Gaussian noise, the least square estimate of the parameter error is given by

$$\underline{\Delta \eta} = W^{+}(t)\ \underline{\Delta x} \qquad (C.10)$$

where

$$W^{+}(t) = [\ W^T W\ ]^{-1}\ W^T\ . \qquad (C.11)$$

For the above inverse to exist, the number of uncertain parameters cannot be greater than the dimension of the state, i.e. $m < n$. When $m > n$, the technique described in this section for obtaining a feedback control is not applicable. Thus, using (C.10) in (C.4), the control becomes

$$\underline{\Delta u} = R^{-1} B^T\ [\ P_1 + P_2 W^{+}\ ]\ \underline{\Delta x}\quad . \qquad (C.12)$$

The least square (LS) feedback gain is therefore

$$K_L(t) = R^{-1}B^T [P_1 + P_2W^+] \tag{C.13}$$

where P_1, P_2 and W^+ are determined from the nonlinear system of equations defined by (C.5), (C.6), (C.9) and (C.11). From (C.12), the LS feedback control is

$$\underline{u}(t,x) = \underline{u}_L(t) + K_L(t)\, \underline{x}(t)$$

with (C.14)

$$\underline{u}_L(t) = \underline{u}_n(t) - K_L(t)\, \underline{x}_n(t) \quad .$$

C.3 Comparison with ...imum Sensitive Feedback Controller

In order to compare the least square feedback gain with that derived in Chapter 4, it will initially be assumed that the number of uncertain parameters is equal to the dimension of the state (m = n). The results of sections 4.3 and 4.4 then apply when the cost functional is given by (4.5) with F_1 replaced by F_2 and with the same weighting matrices as (C.3). In particular, for each component parameter error $\underline{\Delta\eta}_i$ and corresponding column of $[\, {}^{\partial f}/\partial\eta \,]_i \triangleq \underline{g}_i(t)$, $i = 1,\ldots,m$, the MS gain satisfies (from (4.15) - (4.17)),

$$- RK_m\, \underline{s}_i + B^T\, \underline{p}_i = 0$$

with $i = 1,\ldots,m$ and

$$\dot{\underline{s}}_i = A\underline{s}_i + BR^{-1}B^T\, \underline{p}_i + \underline{g}_i \quad ; \quad \underline{s}(0) = 0$$

$$\dot{\underline{p}}_i = -A^T\, \underline{p}_i + Q\, \underline{s}_i \quad ; \quad \underline{p}_i(T) = -D\, \underline{s}_i(T) \quad .$$

These can be combined as in section (4.4) to yield

$$-R\, K_m\, S + B^T\, P = 0 \tag{C.15}$$

where the (nxn) matrices S and P satisfy

$$\dot{S} = AS + HP + G \quad ; \quad S(0) = 0 \quad , \tag{C.16}$$

$$\dot{P} = - A^T P + Q S \quad ; \quad P(T) = - D S(T) \quad , \tag{C.17}$$

with H and G as previously defined. When S(t) is invertable, the minimum sensitive (MS) gain is given by

$$K_m = R^{-1} B^T P S^{-1} . \tag{C.18}$$

The remaining problem is to correlate the MS gain (C.18) with the LS gain given by (C.13). Since m = n, $W^+(t) = W^{-1}(t)$ and (C.13) can be written in the form

$$K_L(t) = R^{-1} B^T [P_1 W + P_2] W^{-1} . \tag{C.19}$$

Note that this is only valid for m = n. When m < n, $W W^+ \neq I$ in general and (C.19) doesn't hold. Let

$$P_L = P_1 W + P_2 \quad . \tag{C.20}$$

Then from (C.9)

$$\dot{W} = AW + H P_L + G \quad ; \quad W(0) = 0 \quad , \tag{C.21}$$

which corresponds in form to (C.16). By differentiating (C.20) and using (C.5), (C.6) and (C.21) the following holds

$$\dot{P}_L = - A^T P_L + Q W \quad ; \quad P_L(T) = - D W(T). \tag{C.22}$$

Thus from (C.19) and (C.20) the LS gain becomes

$$K_L(t) = R^{-1} B^T P_L W^{-1} . \tag{C.23}$$

It is easily seen that by corresponding P_L and W with P and S respectively, the MS and LS feedback gains have similar structures when m = n. However, the linear structure of defining equations is a direct

result of the solution for the MS problem whereas some manipulation is required to obtain this form for the LS solution. The type of least square parameter estimator described in this appendix has been previously used in [28] to obtain neighboring optimum solutions to the cost sensitivity problem, although the existance of the linear solution was not recognized. An additional advantage of the MS gain is that different cost terms corresponding to each parameter error component can be employed as in section 4.4.

The strength of the techniques discussed in Chapter 4 is particularly apparent when $m \neq n$. If $m < n$, the LS gain must be determined from a nonlinear set of matrix differential equations involving a pseudo-inverse. In contrast, the MS gain is determined from a set of linear matrix differential equations resulting from the minimization of a combined state and control cost function (section 4.4). When $m > n$, the LS solution doesn't exist. The MS gain is directly computed from (C.18) with S^{-1} replaced by

$$S^{+} = S^{T} [S S^{T}]^{-1}$$

which exists if the (nxm) matrix S is at least of rank n. It is therefore seen that the results of Chapter 4 apply to a much larger class of problems than does the least square estimator. In addition, the MS gain is always determined from a set of linear differential equations and, because of the variational formulation of the problem, sufficiency conditions can readily be obtained.